Portfolio Selection with a kth-to-default Credit-Linked Note*

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Abstract

We consider a portfolio selection problem of a CRRA investor who faces a risk-free bond and a kth-to-default Credit-Linked Note (CLN). In addition to multiple default protections, the CLN may have both internal and external contagion risks, and its dynamics is obtained under a Markov chain model. By the dynamical programming principle, we characterize the value function as a unique classic solution to a system of Hamilton-Jacobi-Bellman equations, each of which is associated with a default or shock realization state. The optimal strategy is to make the current marginal value of wealth equal the weighted average of the risk-adjusted marginal value of wealth conditional on a default or shock realization, where the weight is determined jointly by the jump size and intensity of CLN. When all reference entities have the same characteristics and the external contagion risk is absent, we prove that the investor will take long/short positions in the CLN if the default risk compensation is positive/negative. Numerically, we find that for a short investment horizon, an additional default protection leads to more investments in the CLN. However, for a long investment horizon, the CLN's early termination compensation becomes more important and may make additional default protection less attractive. This difference between short and long horizons is more salient in the presence of internal and/or external contagion risks.

Keywords: kth-to-default CLN, contagion risk, portfolio optimization

JEL Classification: G11, G13, G31

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1 Introduction

Merton (1969) pioneers the dynamic portfolio selection problems in continuous time. Since then, a large body of literature has been devoted to various extensions of Merton's model. For example, for transaction cost problems, see Davis and Norman (1990); Shreve and Soner (1994); Dai and Yi (2009); for tax problems, see Tahar, Soner, and Touzi (2007, 2010); Dai, Liu, Yang, and Zhong (2015); Cai, Chen, and Dai (2017). Most studies have been focused on the equity market but not the fixed-income market. According to the report "Mapping Global Capital Markets 2011" released by the McKinsey Global Institute, the fixed-income market is almost twice the capitalization of the equity market. Moreover, the data from the National Association of Financial Market Institutional Investor (NAFMII) shows that asset managers are the main purchasers of fixed-income securities. A Credit-Linked Note (CLN) is structured as an ordinary fixed-income security embedded with a Credit Default Swap (CDS), so its access threshold is relatively low. As a result, CLNs are very popular in the fixed-income market and are the second-largest credit derivatives in trading volume after CDS. In this paper, we consider a portfolio selection problem for a constant relative risk aversion (CRRA) investor who allocates her wealth between a risk-free bond and a kth-to-default CLN to maximize her expected utility from the terminal wealth. Our main contributions are threefold as follows.

First, we highlight the *multiple-default* protection feature of a kth-to-default CLN as well as *internal* contagion risks in the reference pool and *external* contagion risks. To the best of our knowledge, it is the first time to add a CLN with these three features to a portfolio selection problem. Following Bo and Capponi (2016), we use a continuous time Markov chain to model the external shock realization and internal default states. This Markovian nature leads to an explicit expression for the market value of the CLN obtained as solutions to Feynman-Kac equations. By a change of measure, we further obtain the dynamics of CLN in the physical measure.

Second, by the dynamical programming principle, we characterize the value function of the portfolio selection problem as a unique classic solution to a system of Hamilton-Jacobi-Bellman (HJB) equations, each of which is associated with a default or shock realization state. The optimal investment strategy in the CLN is to make the current marginal value of wealth equal the weighted average of the risk-adjusted marginal value of wealth conditional on a default or shock realization, where the weight is determined jointly by the jump size and intensity of CLN. In particular, when all reference entities have identical characteristics and the external contagion risk is absent, we prove that the investor will take long/short positions in the CLN if the default risk compensation is positive/negative.

Third, our numerical analysis shows that multiple-default protection has a significant impact on investment strategies. More precisely, multiple-default protection can generate a *non-monotonic* investment strategy in the CLN with respect to the investment horizon, which is in sharp contrast to the *monotonically decreasing* pattern of the investment strategy when there is only one-default protection as presented in Bielecki and Jang (2006). For a short investment horizon, additional default protection leads to more investments in the CLN. However, for a long investment horizon, the CLN's early termination compensation becomes more important and may make additional default protection less attractive. This difference between short and long horizons is more salient in the presence of internal and/or external contagion risks. Moreover, to further quantify the value of additional default protection, we introduce a *proportional certainty equivalent wealth*. On one hand, for ten reference entities, the value of additional default protection drops quickly from over 18% of initial wealth to almost zero as the number of protections increases from 2 to 4. On the other hand, the second-default protection value increases from 1% to over 18% of initial wealth when the number of reference entities increases from 2 to 10.

Related Literature. The most existing literature studying fixed-income portfolio selection problems has focused on securities with only one-default protection, i.e., a security will be terminated once a default event happens. One strand of this literature is devoted to the portfolio selection problems with one or multiple defaultable bonds. For example, Bielecki and Jang (2006) derive optimal investment strategies for a CRRA investor, allocating her wealth among a defaultable bond, a risk-free bank account, and a stock. Kraft and Steffensen (2008) consider an investor who can allocate her wealth across multiple defaultable bonds. Other related works include Bielecki, Jeanblanc, and Rutkowski (2008); Bo, Wang, and Yang (2010); Capponi and Figueroa-Lopez (2014) and references therein. Recently studies have been extended to include credit derivatives. Giesecke, Kim, Kim, and Tsoukalas (2014) study a static selection problem of credit swaps portfolios to maximize an investor's mark-to-market value of the portfolio. Bo and Capponi (2016, 2017) consider the optimal portfolio problem of an investor who wishes to allocate her wealth between several credit default swaps and a money market account. Bo and Ceci (2020) considered an investor who, investing in a credit default swap with the counterparty risk, wants to protect herself against the loss incurred at the default of the counterparty. Since a CLN is structured as a bond embedded with a CDS, our model investigates these two markets in a unified way. More important and different from the above studies, our focus is on the impact of multiple-default protection on optimal investment strategies. In the absence of derivative markets, the multiple defaults are also studied in Jiao, Kharroubi, and Pham (2013), where, using the density hypothesis, the existence and uniqueness of the value function are established via recursive systems of backward stochastic differential equations. In contrast, our model is solved by a system of HJB equations.

Our paper is also related to the studies of contagion risks. Callegaro, Jeanblanc, and Runggaldier (2012) consider the problem of maximization of expected utility from terminal wealth with contagion risks under incomplete information. Bo and Capponi (2016) develop a dynamic portfolio optimization framework for credit derivatives with interacting default intensities via a continuous-time Markov chain model. Consistent with the empirical findings in Azizpour, Giesecke, and Schwenkler (2018), Bo, Capponi, and Chen (2019) developed a fixed-income portfolio framework to capture the exponential decay of contagious intensities between successive default events. The above literature examines the impact of contagion risks on multiple defaultable bonds or credit derivatives. We complement their studies by focusing on the impact of contagion risks on the kth-to-default CLN, which is a more complicated and hybrid fixed-income security.

The rest of the paper is organized as follows. In Section 2, we first briefly introduce the CLN. Then, the CLN's value and its dynamics are derived by using a continuous Markov chain model. Section 3 is devoted to the setup of the portfolio selection problem between a risk-free bond and a kth-to-default CLN. Theoretical analysis of the model is presented in Section 4 for the general case, and Section 5 for the case with identical reference entities, respectively. Section 6 turns to a numerical analysis. We provide a short conclusion in Section 7.

2 The Model

In this section, we first briefly introduce Credit-Linked Notes. Then, a default model is proposed by using a continuous-time Markov chain. We next derive the dynamics of a kth-to-default CLN with both *internal* and *external* contagion risks.

2.1 Credit-Linked Notes

We consider a kth-to-default CLN with N entities in the reference pool and a finite maturity $\overline{T} < \infty$, where $N \geq k \geq 1$. The note consists of three parts: N reference entities, a CLN issuer, and an investor. It allows the protection buyer to transfer specific credit risk to the protection seller.

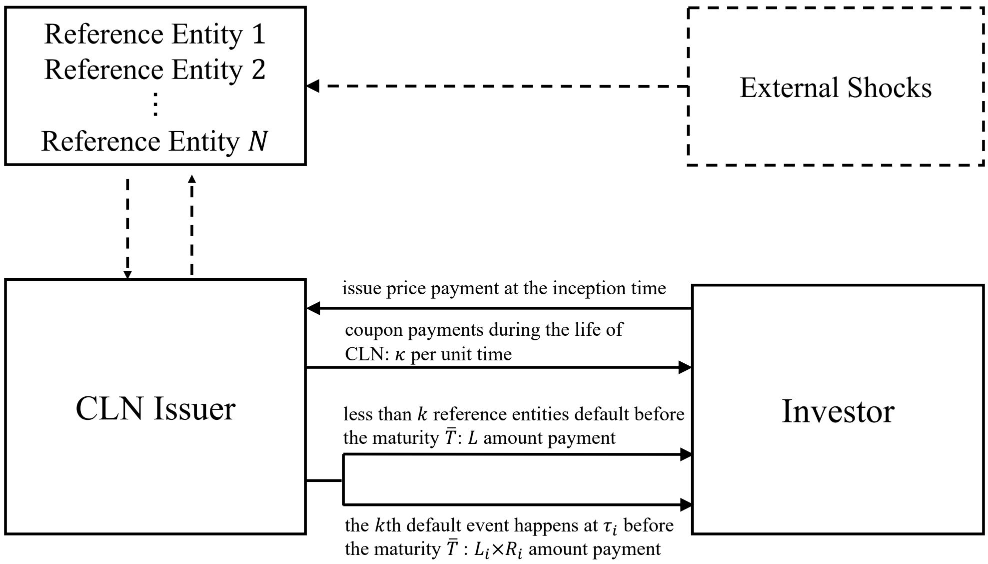

Figure 1: The cash flow of a kth-to-default CLN.

Figure 1 illustrates the cash flow of a kth-to-default CLN. More specifically, the investor pays an issue price of the CLN at an inception time, and then receives regular coupon payments ($\kappa > 0$ per unit time) from the issuer during the life of the CLN. The CLN will be terminated before the maturity if and only if the kth default in the reference pool occurs prior to the maturity, at which time the investor gets only part of the nominal principal of the CLN. For example, suppose that the ith ($i = 1, 2, \cdots, N$) reference entity defaults at time $\tau_i \leq \overline{T}$, and there are already other $k - 1$ reference entities defaulted before. Then, at time τ_i, the CLN must be terminated immediately and the investor receives amount of $R_i \times L_i$ payment, where $R_i \in [0, 1]$ and $L_i > 0$ are recovery rate and the nominal principal for the ith reference entity, respectively. On the other hand, if the early termination does not happen, the investor receives the full nominal principal ($L > 0$) of the CLN at the maturity $\overline{T}$. In sum, this note can be viewed as an ordinary fixed-income security embedded with a kth-to-default CDS.

To characterize the defaults in the reference pool. Roughly speaking, two kinds of risks are considered in the literature. One is called the *internal* contagion risk. That is, a default of one individual reference entity may change the default intensities of others in the reference pool. The other is known as the *external* contagion risk typically arsing from market-wide shocks in a specific period such as the financial crisis in 2007-2008. A realization of the external shock can also alter the intensities of reference entities. In this paper, we consider both internal and external contagion risks and their impacts on the optimal portfolio strategies later in Section 3. To model a rare disaster event in a parsimonious way, we only study a one-time external shock. Next, we introduce the shock realization and default model.

2.2 The Shock Realization and Default Model

Following Bélanger, Shreve, and Wong (2004) and Bo and Capponi (2016), we use a $(N+1)$-dimensional indicator process $\mathbf{H}(t) = (H_0(t), H_1(t), \cdots, H_N(t))$ to describe the realization and default state. $H_0(t) = \mathbf{1}_{\{\tau_0 \le t\}}$ is the realization state of the one-time external shock, where τ_0 is the realization time of the shock:

$$\tau_0 = \inf\{t \ge 0; H_0(t) = 1\} \ .$$

Here $\mathbf{1}_{\{A\}}$ is an indicator function of a set A, which equals 1 if $A \neq \emptyset$ and 0 otherwise. Meanwhile, $H_i(t) = \mathbf{1}_{\{\tau_i \le t\}}$ is the default state of the ith reference entity, where τ_i is the default time defined as

$$\tau_i = \inf\{t \ge 0; H_i(t) = 1\}, \quad i \in I = \{1, 2, \cdots, N\}. \tag{1}$$

The indicator process is supported by a filtered probability space $(\Omega, \mathcal{G}, \{\mathcal{G}_t\}_{t\ge 0}, \mathbb{Q})$, where $\mathcal{G}_t = \sigma(\mathbf{H}(s); s \le t)$, and $\mathbb{Q}$ denotes the risk-neutral probability measure. We also denote by $\mathbb{G} = (\mathcal{G}_t, t \ge 0)$ the market information filtration. Let $\mathcal{S} = \{0,1\}^{N+1}$ be the state space of the indicator process $\mathbf{H} = (\mathbf{H}(t); t \ge 0)$. As in Bo and Capponi (2016), we assume that both shock realization and defaults of reference entities cannot happen simultaneously.

We assume that $\mathbf{H}(t)$ follows a *continuous-time Markov chain* on $\mathcal{S}$ with a transition rate $(1 - H_i(t))h_i(\mathbf{H}(t))$ to its neighboring state $\mathbf{H}^i(t) := (H_0(t), \cdots, H_{i-1}(t), 1 - H_i(t), H_{i+1}(t), \cdots, H_N(t))$ for $i \in I \cup \{0\}$. Here $h_i(\mathbf{z})$ is a positive measurable function defined on $\mathbf{z} = (z_0, z_1, \cdots, z_N) \in \mathcal{S}$ such that

$$\xi_i(t) := H_i(t) - \int_0^t (1 - H_i(s))h_i(\mathbf{H}(s))ds, \quad t \ge 0 \tag{2}$$

is a $(\mathbb{Q}, \mathcal{G}_t)$-martingale. For the external shock, we assume that its intensity is constant all the time, i.e.,

$$h_0(\mathbf{z}) = a_0 > 0 \ .$$

In contrast, to capture both *internal* and *external* contagion risks, we further assume that $h_i(\cdot)$ admits the following form (Leung and Yue, 2009):

$$h_i(\mathbf{z}) = a_i\big((\alpha_{0,i} - 1)z_0 + 1\big) \prod_{j \in I \setminus \{i\}} \big((\alpha_{j,i} - 1)z_j + 1\big), \quad i \in I, \tag{3}$$

where $a_i > 0$ is the base default intensity of the ith reference entity, $\alpha_{0,i} > 0$ is the external contagious factor from the external shock to the ith reference entity, and $\alpha_{j,i} > 0$ is the internal contagious factor from the jth reference entity to the ith for $i, j \in I$. The interpretation of (3) is as follows. If defaults never happen in the reference pool and the external shock does not come, the base default intensity of each individual reference entity is a positive constant, i.e., $h_i(\mathbf{0}) = a_i$ for all $i \in I$. If some reference entity defaults, this individual default risk will have a contagious impact on other reference entities by altering their default intensities permanently. For example, if the jth reference entity defaults, the default intensity of the ith reference entity will have a *proportional* jump size $\alpha_{j,i}$ for all $i \in I \setminus \{j\}$. Depending on whether or not $\alpha_{j,i}$ is greater than 1, the intensity $h_i(\mathbf{z})$ can jump upward or downward accordingly. Similarly, if the external shock is realized, the intensity of every reference entity in the pool will jump by a proportion equaling $\alpha_{0,i}$ for all $i \in I$.

Since the early termination event happens if and only if there are k default times before the CLN's maturity, it is useful to introduce the following default index:

Definition 1 (Default Index)**.** *Let $M : \mathbf{z} \in \mathcal{S} \mapsto M(\mathbf{z}) \subset I$ be the index set of the reference entities that have defaulted. The norm of M, denoted by $|M|$, is defined as the number of the reference entities that have defaulted. Consequently, $I \setminus M(\mathbf{z})$ is the index set of the reference entities that have not defaulted.*

Remark 2.1. *The default index only records the default state in the reference pool, so it is not affected by the realization of the external shock. For example, suppose that there are $N = 5$ reference entities (i.e., $I = \{1, 2, 3, 4, 5\}$), and that the second and fifth ones have defaulted at time t. If the external shock is not realized (i.e., $z_0 = 0$), then $\mathbf{H}(t) = \mathbf{z} = (0, 0, 1, 0, 0, 1)$. Otherwise, if the external shock is realized (i.e., $z_0 = 1$), then $\mathbf{H}(t) = \mathbf{z} = (1, 0, 1, 0, 0, 1)$. For both cases, Definition 1 implies that $M(\mathbf{z}) = \{2, 5\}$,*

$|M(\mathbf{z})| = 2$, and $I \backslash M(\mathbf{z}) = \{1, 3, 4\}$. Another example is the initial state where no realization and defaults happen, which means $\mathbf{H}(t) = \mathbf{0} = (0, 0, 0, 0, 0, 0)$. In this case, $M(\mathbf{0}) = \emptyset$, $|M(\mathbf{0})| = 0$, and $I \backslash M(\mathbf{0}) = I = \{1, 2, 3, 4, 5\}$.

We also introduce the set $\mathcal{Z}_k$:

$$\mathcal{Z}_k = \{\mathbf{z} \in \mathcal{S} \, : \, |M(\mathbf{z})| < k\}, \tag{4}$$

which collects the state where the cumulative default times in the reference pool is less than k. Now, we can formally defined the critical default time denoted by τ, the kth default time in the reference pool, as the first exit time of the set $\mathcal{Z}_k$ for the indicator process $\mathbf{H}$ with $M(H(t-)) \in \mathcal{Z}_k$ at a given time $t < \overline{T}$:

$$\tau = \inf\{s \geq t \, : \, M(\mathbf{H}(s)) \notin \mathcal{Z}_k, \; M(\mathbf{H}(t-)) \in \mathcal{Z}_k\} \tag{5}$$

with the convention that $\inf\{\emptyset\} = \infty$. As a result, the kth-to-default CLN will be terminated at the time $\min(\tau, \overline{T})$.

2.3 The Market

We consider a financial market consisting of a risk-free bond with a constant interest rate $r > 0$ and a kth-to-default CLN introduced in Section 2.1. Without loss of generality, we normalize the nominal principals to be one dollar as in the literature, i.e., $L_i = L = 1$.

Next, to examine the dynamics of the CLN, consider first the pre-termination value of the CLN at time t denoted by $C(t, \mathbf{H}(t))$. That is, the current time t is strictly less than the critical time τ defined in (5), or equivalently $\mathbf{H}(t) = \mathbf{z} \in \mathcal{Z}_k$. Then, the no-arbitrage argument implies that

$$C(t, \mathbf{H}(t)) = C^{\kappa}(t, \mathbf{H}(t)) + \sum_{i \in I \backslash M(\mathbf{H}(t))} C^i(t, \mathbf{H}(t)) + C^L(t, \mathbf{H}(t)) \;, \tag{6}$$

where functions $C^i(t, \mathbf{z})$, $C^{\kappa}(t, \mathbf{z})$, and $C^L(t, \mathbf{z})$ are given respectively by

$$C^{\kappa}(t, \mathbf{z}) = \mathbb{E}\Big[\int_t^{\min(\tau, \overline{T})} e^{-r(u-t)} \kappa \, du \,\Big|\, \mathcal{G}_t\Big] \;, \tag{7}$$

$$C^i(t, \mathbf{z}) = \mathbb{E}\Big[e^{-r(\tau_i - t)} \mathbf{1}_{\{\tau = \tau_i \leq \overline{T}\}} R_i \,\Big|\, \mathcal{G}_t\Big] \;, \tag{8}$$

$$C^L(t, \mathbf{z}) = \mathbb{E}\Big[e^{-r(\overline{T} - t)} \mathbf{1}_{\{\overline{T} < \tau\}} \,\Big|\, \mathcal{G}_t\Big] \;. \tag{9}$$

Here τ_i defined in (1) and τ defined in (5) are the default time of the ith reference entity and the kth default time in the reference pool, respectively. Their economic meanings are as follows. $C^\kappa(t,\mathbf{z})$ is the present value of receiving κ per unit time premium during the life of CLN. $C^i(t,\mathbf{z})$ stands for the present value of the payment R_i when the critical kth default happens on the ith reference entity before the maturity $\overline{T}$. Finally, $C^L(t,\mathbf{z})$ gives the present value of getting one dollar nominal principal at the maturity of CLN when there is no early termination.

By the Feynman-Kac formula, $C^\kappa(\,\cdot\,)$, $C^i(\,\cdot\,)$, and $C^L(\,\cdot\,)$ respectively satisfy the following differential equations: For $(t,\mathbf{z}) \in [0,T) \times \mathcal{Z}_k$,

$$\Big(\frac{\partial}{\partial t} + \mathcal{A}\Big) C^\kappa(t,\mathbf{z}) + \kappa \;=\; r\, C^\kappa(t,\mathbf{z}), \tag{10}$$

$$\Big(\frac{\partial}{\partial t} + \mathcal{A}\Big) C^i(t,\mathbf{z}) \;=\; r\, C^i(t,\mathbf{z}), \tag{11}$$

$$\Big(\frac{\partial}{\partial t} + \mathcal{A}\Big) C^L(t,\mathbf{z}) \;=\; r\, C^L(t,\mathbf{z}), \tag{12}$$

where the operator $\mathcal{A}$ is the infinitesimal generator of the indicator process $\mathbf{H}$:

$$\mathcal{A}\, g(\mathbf{z}) = \sum_{j\in I\cup\{0\}} \big(g(\mathbf{z}^j) - g(\mathbf{z})\big)(1-z_j)h_j(\mathbf{z}) = \sum_{j\in I\cup\{z_0\}\setminus M(\mathbf{z})} \big(g(\mathbf{z}^j) - g(\mathbf{z})\big)h_j(\mathbf{z})$$

for any measurable function $g(\mathbf{z})$ defined on $\mathbf{z} \in \mathcal{S}$. Here the set $I \setminus M(\mathbf{z})$ stands for the non-default indices of the reference entities. If the external shock is not realized (i.e., $z_0 = 0$), then $I \cup \{z_0\} \setminus M(\mathbf{z}) = I \cup \{0\} \setminus M(\mathbf{z})$, which includes the indices of non-default entities and unrealized shock. In contrast, if the external shock is realized (i.e., $z_0 = 1$), then $I \cup \{z_0\} \setminus M(\mathbf{z}) = I \setminus M(\mathbf{z})$, which contains only the non-default indices of the reference entities.

The following proposition shows that the pre-termination value function $C(t,\mathbf{z})$ is continuously differentiable in $t \in [0,\overline{T}]$ for all $\mathbf{z} \in \mathcal{Z}_k$.

Proposition 1. *For all $\mathbf{z} \in \mathcal{Z}_k$, the functions $C^\kappa(t,\mathbf{z})$, $C^i(t,\mathbf{z})$, and $C^L(t,\mathbf{z})$ are continuously differentiable in $t \in [0,\overline{T}]$. Consequently, $C(t,\mathbf{z})$ given by (6) is also continuously differentiable in $t \in [0,\overline{T}]$ for all $\mathbf{z} \in \mathcal{Z}_k$. Moreover, $C(t,\mathbf{z})$ admits an explicit form given in Appendix B.*

Proof. First, from (7) and (9), it is easy to check the following boundary and terminal conditions hold:

$$C^\kappa(\overline{T},\mathbf{z}) = 0 \;\text{ for } z \in \mathcal{Z}_k, \qquad C^\kappa(t,\mathbf{z}) = 0 \;\text{ for } t \in [0,\overline{T}),\; \mathbf{z} \in \partial\mathcal{Z}_k\ ,$$

$$C^L(\overline{T},\mathbf{z}) = 1 \;\text{ for } z \in \mathcal{Z}_k, \qquad C^L(t,\mathbf{z}) = 0 \;\text{ for } t \in [0,\overline{T}),\; \mathbf{z} \in \partial\mathcal{Z}_k\ .$$

Consequently, the classical theory of the linear ordinary differential equations (ODEs) implies that the differential equations (10) and (12) combined with the above conditions have unique classical solutions for all $\mathbf{z} \in \mathcal{Z}_k$.

Next, we study the function $C^i(t, \mathbf{z})$ for $i \in I$. On one hand, the definition (8) implies that the following terminal condition hold:

$$C^i(\overline{T}, \mathbf{z}) = 0 \ . \tag{13}$$

On the other hand, as there are k default protections, we first consider the case of $|M(z)| = k - 1$. That is, there is only one default protection. In this case, we also have two subcases: (i) $z_i = 1$, and (ii) $z_i = 0$.

(i) when $z_i = 1$, which means the ith reference entity has defaulted already. From the definition (8), we must have

$$C^i(t, \mathbf{z}) = 0 \quad \text{for } t \in [0, \overline{T}) \ .$$

(ii) when $z_i = 0$, which means the ith reference entity is still alive. Then, the definition (8) implies that

$$C^i(t, \mathbf{z}^i) = R_i, \ \text{and } C^i(t, \mathbf{z}^j) = 0 \quad \text{for all } j \in I \cup \{z_0\} \setminus M(\mathbf{z}) \setminus \{i\}, \ t \in [0, \overline{T}) \ .$$

Plugging the above conditions into equation (11) yields

$$\frac{dC^i(t, \mathbf{z})}{dt} - \Big(r + \sum_{j \in I \cup \{z_0\} \setminus M(\mathbf{z}) \setminus \{i\}} h_j(\mathbf{z})\Big) C^i(t, \mathbf{z}) = -R_i h_i(\mathbf{z}) \ .$$

Hence, this ODE together with the terminal condition (13) also has a unique classical solution.

Finally, for the case of $|M(\mathbf{z})| < k-1$, we can use the terminal condition (13) plus the boundary condition given in the case of $|M(\mathbf{z})| = k - 1$ to derive the required solutions. This completes the proof. $\square$

Once the pre-termination value $C(t, \mathbf{H}(t))$ is known, the value of the kth-to-default CLN, denoted by $\widetilde{C}(t, \mathbf{H}(t))$, is clearly given by

$$\widetilde{C}(t, \mathbf{H}(t)) = \mathbf{1}_{\{t<\tau\}} C(t, \mathbf{H}(t)) = \mathbf{1}_{\{|M(\mathbf{H}(t))| \le k-1\}} C(t, \mathbf{H}(t)) \ .$$

The following proposition characterizes the dynamics of $\widetilde{C}(t, \mathbf{H}(t))$ under the risk-neutral measure $\mathbb{Q}$, whose proof is relegated in Appendix C.

Proposition 2. *The $\mathbb{Q}$-dynamics of the kth-to-default CLN value is given by*

$$\frac{d\widetilde{C}(t, \mathbf{H}(t))}{\widetilde{C}(t, \mathbf{H}(t-))}$$

$$
\begin{aligned}
&=\Bigg\{\Big(r-\frac{\kappa}{C(t,\mathbf{H}(t-))}\Big)dt+\sum_{j\in I\setminus M(\mathbf{H}(t-))}\Big(\frac{C(t,\mathbf{H}^j(t-))}{C(t,\mathbf{H}(t-))}-1\Big)d\xi_j(t)\Bigg\}\mathbf{1}_{\{|M(\mathbf{H}(t-))|<k-1\}}\\
&\quad+\Bigg\{\Big(r-\frac{\kappa+\sum_{j\in I\setminus M(\mathbf{H}(t-))}R_jh_j(\mathbf{H}(t-))}{C(t,\mathbf{H}(t-))}\Big)dt-\sum_{j\in I\setminus M(\mathbf{H}(t-))}d\xi_j(t))\Bigg\}\mathbf{1}_{\{|M(\mathbf{H}(t-))|=k-1\}}\\
&\quad+\Bigg\{(1-H_0(t-))\Big(\frac{C(t,\mathbf{H}^0(t-))}{C(t,\mathbf{H}(t-))}-1\Big)d\xi_0(t)\Bigg\}\mathbf{1}_{\{|M(\mathbf{H}(t-))|\le k-1\}}\ ,
\end{aligned}
\tag{14}
$$

with $\widetilde{C}(0,\mathbf{H}(0-))=C(0,\mathbf{H}(0-))$.

From the above dynamics, it is important that a default event has different impacts on both drift and jump terms when $|M(\mathbf{H}(t-))|<k-1$ or $|M(\mathbf{H}(t-))|=k-1$. For the drift term, since a default will not trigger an early termination when $|M(\mathbf{H}(t-))|<k-1$, there are no expected cash payments except for an ordinary coupon payment (κ per unit). As a result, the default has no direct impact on the drift term. In contrast, expected payments will be received due to the early termination triggered by a default event when $|M(\mathbf{H}(t-))|=k-1$. This explains why there is an extra term $\sum_{j\in I\setminus M(\mathbf{H}(t-))}R_jh_j(\mathbf{H}(t-))/C(t,\mathbf{H}(t-))$ in the drift. For the jump term, when $|M(\mathbf{H}(t-))|=k-1$, the proportional jump size is -1 so that $\widetilde{C}(t,\mathbf{H}(t))=0$. This is because the CLN is terminated at the kth default time. In contrast, when $|M(\mathbf{H}(t-))|<k-1$, the CLN is still alive even if there were a default. As a result, the proportional jump size $C(t,\mathbf{H}^j(t-))/C(t,\mathbf{H}(t-))-1$ is strictly large than -1 for $j\in I\setminus M(\mathbf{H}(t-))$. Finally, the last term in (14) arises from the presence of the external shock. Different from the default risk, the external shock has no impact on the drift because it does not generate any cash flow. However, there will be a proportional jump equaling $C(t,\mathbf{H}^0(t-))/C(t,\mathbf{H}(t-))-1$ when the external shock is realized, which triggers intensity jumps in the reference pool.

Next, we derive the dynamics of CLN in the objective measure $\mathbb{P}$ as the investor wishes to optimize his utility from terminal wealth under the real-world measure. However, the value observed in the market is given under the risk-neutral measure $\mathbb{Q}$. To this purpose, let $\lambda_0(\mathbf{z})=\lambda_0\in(-1,\infty)$ be a constant, and let $\lambda_i(\mathbf{z})\in(-1,+\infty)$ be an arbitrary bounded measurable function defined on $\mathbf{z}\in\mathcal{S}$ for $i\in I$. Assume that the process $X=(X(t);t\ge0)$ satisfies the following SDE given by

$$
\frac{dX(t)}{X(t-)}=\sum_{i=0}^{N}\lambda_i(\mathbf{H}(t-))d\xi_i(t),\quad X(0-)=1\ ,
$$

where $\xi_i=(\xi_i(t);t\ge0)$ is defined by (2). Then we have the following result for the change of measure, whose proof can be found in Bo and Capponi (2016).

Lemma 2.1. *For $T > 0$, define a new probability measure $\mathbb{P} \ll \mathbb{Q}$ on $\mathcal{G}_T$ by*

$$d\mathbb{P} = X(T)d\mathbb{Q} \ ,$$

then, for all $i \in I \cup \{0\}$,

$$\xi_i^{\mathbb{P}}(t) = H_i(t) - \int_0^t (1 - H_i(s))h_i^{\mathbb{P}}(\mathbf{H}(s))ds, \ t \geq 0$$

is a $(\mathbb{P}, \mathcal{G}_t)$-martingale. $h_i^{\mathbb{P}}(\mathbf{H})$ is the $\mathbb{P}$-default intensity of the reference entity i, and satisifies

$$h_i^{\mathbb{P}}(\mathbf{z}) = \big(1 + \lambda_i(\mathbf{z})\big)h_i(\mathbf{z}), \quad \mathbf{z} \in \mathcal{S} \ . \tag{15}$$

With the help of Lemma 2.1, we can derive the $\mathbb{P}$-dynamics of the CLN as follows.

Proposition 3. *The $\mathbb{P}$-dynamics of the kth-to-default CLN is given by*

$$\begin{aligned}
&\frac{d\widetilde{C}(t, \mathbf{H}(t))}{\widetilde{C}(t, \mathbf{H}(t-))} \\
=&\Bigg\{ \Bigg[r - \frac{\kappa}{C(t, \mathbf{H}(t-))} + \sum_{j \in I \setminus M(\mathbf{H}(t-))} \Big(\frac{C(t, \mathbf{H}^j(t-))}{C(t, \mathbf{H}(t-))} - 1 \Big) \lambda_j(\mathbf{H}(t-))h_j(\mathbf{H}(t-)) \Bigg] dt \\
&\quad + \sum_{j \in I \setminus M(\mathbf{H}(t-))} \Big(\frac{C(t, \mathbf{H}^j(t-))}{C(t, \mathbf{H}(t-))} - 1 \Big) d\xi_j^{\mathbb{P}}(t) \Bigg\} \mathbf{1}_{\{|M(\mathbf{H}(t-))| < k-1\}} \\
&+ \Bigg\{ \Bigg[r - \frac{\kappa + \sum_{j \in I \setminus M(\mathbf{H}(t-))} R_j h_j(\mathbf{H}(t-))}{C(t, \mathbf{H}(t-))} - \sum_{j \in I \setminus M(\mathbf{H}(t-))} \lambda_j(\mathbf{H}(t-))h_j(\mathbf{H}(t-)) \Bigg] dt \\
&\quad - \sum_{j \in I \setminus M(\mathbf{H}(t-))} d\xi_j^{\mathbb{P}}(t) \Bigg\} \mathbf{1}_{\{|M(\mathbf{H}(t-))| = k-1\}} \\
&+ \Bigg\{ (1 - H_0(t-)) \Big(\frac{C(t, \mathbf{H}^0(t-))}{C(t, \mathbf{H}(t-))} - 1 \Big) \Big(\lambda_0 \, a_0 dt + d\xi_0^{\mathbb{P}}(t) \Big) \Bigg\} \mathbf{1}_{\{|M(\mathbf{H}(t-))| \leq k-1\}} \ .
\end{aligned} \tag{16}$$

with $\widetilde{C}(0, \mathbf{H}(0-)) = C(0, \mathbf{H}(0-))$.

In the literature, $\lambda_i(\mathbf{z})$ is related to the *default risk premium* of the ith reference entity. This terminology is an analog to the market price of risk. Intuitively, the investor needs suitable risk compensations to bear the default risks in the objective measure. Hence, the extra terms $\sum_{j \in I \setminus M(\mathbf{H}(t-))} \Big(\frac{C(t, \mathbf{H}^j(t-))}{C(t, \mathbf{H}(t-))} - 1 \Big) \lambda_j(\mathbf{H}(t-))h_j(\mathbf{H}(t-))$ and $-\sum_{j \in I \setminus M(\mathbf{H}(t-))} \lambda_j(\mathbf{H}(t-))h_j(\mathbf{H}(t-))$ account for the compensations for the default risks when $|M(\mathbf{H}(t-))| < k - 1$ and $|M(\mathbf{H}(t-))| = k - 1$, respectively. Finally, the last term $\Big(\frac{C(t, \mathbf{H}^0(t-))}{C(t, \mathbf{H}(t-))} - 1 \Big) \lambda_0 \, a_0$ is the compensation for the external shock.

3 Portfolio Selection Problem

Following Merton (1969), we consider an investor's portfolio selection problem between a riskless bond and a kth-to-default CLN introduced in Section 2 to maximize his expected utility of the wealth at a scheduled terminal time T. Without loss of generality, we assume that the investment horizon T is less than the maturity $\overline{T}$ of the CLN. Next, we introduce the required investment strategies and the associated wealth processes.

3.1 Investment Strategies and Wealth Processes

Let $\phi(t)$ be the number of shares of the kth-to-default CLN that the investor buys ($\phi(t) > 0$) or sells ($\phi(t) < 0$) at time t. Denote by $B(t)$ and $\phi^B(t)$ the price of the riskless bond and the number of shares invested in the bond, respectively. The process $\boldsymbol{\phi} = (\phi(t), \phi^B(t);\ 0 \leq t \leq T)$ is called a portfolio process or a portfolio investment strategy. As usual, we first require the portfolio process $\boldsymbol{\phi}$ to be $\mathbb{G}$-adapted.

Now, given a $\mathbb{G}$-adapted portfolio process $\boldsymbol{\phi} = (\phi(t), \phi^B(t);\ 0 \leq t \leq T)$, its associated wealth process, denoted by $W^{\boldsymbol{\phi}}(t)$, is defined by

$$W^{\boldsymbol{\phi}}(t) = \phi(t)\, \widetilde{C}(t, \mathbf{H}(t)) + \phi^B(t) B(t), \qquad 0 \leq t \leq T.$$

The portfolio process $\boldsymbol{\phi}$ is said to be *self-financing* if $W^{\boldsymbol{\phi}}(t) = W^{\boldsymbol{\phi}}(0-) + G^{\boldsymbol{\phi}}(t)$, where $G^{\boldsymbol{\phi}}(t)$ is the cumulative gain process defined by

$$G^{\boldsymbol{\phi}}(t) = \int_0^t \phi(s-)[d\widetilde{C}(s, \mathbf{H}(s)) + dD(s)] + \int_0^t \phi^B(s) dB(s), \quad 0 \leq t \leq T\ .$$

Here $D = (D(t);\ 0 \leq t \leq T)$ is the cumulative dividend process corresponding to the kth-to-default CLN and satisfies $D(0-) = 0$, and

$$D(t) = \int_0^t \kappa\, \mathbf{1}_{\{s < \min(\tau, \overline{T})\}} ds + \sum_{i \in I} R_i \mathbf{1}_{\{\tau_i = \tau \leq t\}} + \mathbf{1}_{\{t = \overline{T} < \tau\}}\ , \tag{17}$$

where τ defined in (5) is the kth default times in the reference pool. (17) essentially describes the cash flow of the CLN from the issuer to the investor which consists of three parts: (i) a continuous coupon payment κ per unit time during the life of the CLN, (ii) a one-time payment if early termination occurs, and (iii) a one-time payment at the maturity if no early termination.

In this paper, we focus on a class of feedback investment strategies, which are formally defined below:

Definition 2 (Admissible Feedback Investment Strategy)**.** *Given* $(t, w, \mathbf{z}) \in [0, T] \times \mathbb{R}_+ \times \mathcal{S}$*, a portfolio process* $\boldsymbol{\phi} = (\phi(u), \phi^B(u);\ t \le u \le T)$ *is said to be a* $(t, w, \mathbf{z})$*-admissible feedback investment strategy provided that (i)* ϕ *is* $\mathbb{G}$*-predictable and locally bounded, (ii) the associated wealth process* $W^{\boldsymbol{\phi}}$ *starting from* $W(t-) = w$ *is nonnegative, and (iii)* $\phi(u)$ *has the following feedback form:*

$$\phi(u) = \widetilde{\phi}(u, W^{\phi}(u-), \mathbf{H}(u-)), \qquad u \in [t, T] \ ,$$

where $\widetilde{\phi}(\,\cdot\,)$ *is a function defined on* $[0, T] \times \mathbb{R}_+ \times \mathcal{S}$*.*

It turns out that it is convenient to rewrite the investment strategy as the proportion of wealth invested in the CLN defined below:

$$\pi(t) = \frac{\phi(t)\widetilde{C}(t, \mathbf{H}(t))}{W^{\boldsymbol{\phi}}(t)} \ . \tag{18}$$

Therefore, $1 - \pi(t)$ is the proportion of wealth invested in the riskless bond. In the sequel, we will also call $\pi = (\pi(u);\ t \le u \le T)$ defined in (18) a $(t, w, \mathbf{z})$-admissible feedback investment strategy whenever $\boldsymbol{\phi} = (\phi(u), \phi^B(u);\ t \le u \le T)$ is admissible. In addition, we shall write the wealth process $W^{\boldsymbol{\phi}}(t)$ as $W^{\pi}(t)$ or simply $W(t)$ interchangeably.

Proposition 4. *Given* $(w, \mathbf{z}) \in \mathbb{R}_+ \times \mathcal{S}$*, let* $\boldsymbol{\phi} = (\phi(t), \phi^B(t);\ 0 \le t \le T)$ *or equivalently* $\pi = (\pi(t);\ 0 \le t \le T)$ *defined in* (18) *be a* $(0, w, \mathbf{z})$*-admissible feedback investment strategy from the initial time. Then, the* $\mathbb{P}$*-dynamics of the wealth process is given by*

$$\begin{aligned} \frac{dW(t)}{W(t-)} &= \Big(r - \pi(t-) \sum_{j \in I \cup \{z_0\} \setminus M(\mathbf{H}(t-))} \Delta^M\big(t, \mathbf{H}(t-), \mathbf{H}^j(t-)\big)\, h_j(\mathbf{H}(t-))\Big) dt \\ &\quad + \pi(t-) \sum_{j \in I \cup \{z_0\} \setminus M(\mathbf{H}(t-)} \Delta^M\big(t, \mathbf{H}(t-), \mathbf{H}^j(t-)\big)\, dH_j(t) \ , \end{aligned} \tag{19}$$

where

$$\Delta^M(t, \mathbf{z}, \mathbf{z}^j) = \mathbf{1}_{\{|M(\mathbf{z})| \le k-1\}} \left(\frac{C(t, \mathbf{z}^j)}{C(t, \mathbf{z})} - 1 \right) \ . \tag{20}$$

Remark 3.1. *According to* (19)*, a default event triggers a jump in the wealth process such that*

$$W(t) = \Big(1 + \pi(t-) \sum_{j \in I \setminus M(\mathbf{H}(t-))} \Delta^M\big(t, \mathbf{H}(t-), \mathbf{H}^j(t-)\big)\, dH_j(t)\Big) W(t-) \ .$$

When $|M(\mathbf{H}(t-))| \le k - 1$*, if the* j*th reference entity defaults, the wealth will have a jump given by* $W(t) = \big(1 + \pi(t-)\, \mathbf{1}_{\{|M(\mathbf{H}(t-)| \le k-1\}} \big[C(t, \mathbf{H}^j(t-))/C(t, \mathbf{H}(t-)) - 1\big]\big) W(t-)$.

However, when $|M(\mathbf{H}(t-))| \geq k$, $\Delta^M\big(t, \mathbf{H}(t-), \mathbf{H}^j(t-)\big) = 0$ for all j. Consequently, the wealth grows at the risk-free rate r, i.e., $dW(t) = rW(t)dt$. In contrast, a realization of the external shock always leads to a jump in the wealth process such that

$$W(t) = \Big(1 + \pi(t-)(1 - H_0(t-))\Delta^M\big(t, \mathbf{H}(t-), \mathbf{H}^0(t-)\big)\, dH_0(t)\Big)W(t-) \ .$$

From Definition 2, the wealth process $W(t)$ is required positive under an admissible investment strategy $\pi(t)$. According to Jacod and Shiryaev (2003) and the dynamics of $W(t)$ given in (19), the investment strategy has to satisfy the following condition:

$$1 + \pi(t-)\, \Delta^M(t, \mathbf{z}, \mathbf{z}^j) > 0, \quad j \in I \cup \{z_0\} \setminus M(\mathbf{z}) \ . \tag{21}$$

The following proposition characterizes the investment constraint in terms of $\pi(t)$ defined in (18). Thanks to the linear structure of the dynamics (19), the condition (21) implies that the investment constraint, denoted by Π, depends on $(t, \mathbf{z})$ but not on the wealth level w.

Proposition 5 (Investment Constraint)**.** *For all $(t, w, \mathbf{z}) \in [0, T) \times \mathbb{R}_+ \times \mathcal{S}$, an admissible investment strategies $\pi(t)$ must satisfy:*

$$\pi(t) \in \Pi(t, \mathbf{z}) = \big(\underline{\pi}(t, \mathbf{z}),\ \overline{\pi}(t, \mathbf{z})\big) \ ,$$

where the lower and upper bounds $\underline{\pi}(t, \mathbf{z})$ and $\overline{\pi}(t, \mathbf{z})$ are given as follows. If $\mathbf{z} \in \mathcal{S} \setminus \mathcal{Z}_k$, then $\underline{\pi}(t, \mathbf{z}) = -\infty$, and $\overline{\pi}(t, \mathbf{z}) = +\infty$. Otherwise, we have:

(i) If $C(t, \mathbf{z}^j) > C(t, \mathbf{z})$ for all $j \in I \cup \{z_0\} \setminus M(\mathbf{z})$,

$$\underline{\pi}(t, \mathbf{z}) = \max_{j \in I \cup \{z_0\} \setminus M(\mathbf{z})} \left\{ -\frac{C(t, \mathbf{z})}{C(t, \mathbf{z}^j) - C(t, \mathbf{z})} \right\}, \quad \overline{\pi}(t, \mathbf{z}) = +\infty \ ,$$

where function $C(\,\cdot\,)$ is the pre-termination value of the CLN.

(ii) If $C(t, \mathbf{z}^j) < C(t, \mathbf{z})$ for all $j \in I \cup \{z_0\} \setminus M(\mathbf{z})$,

$$\underline{\pi}(t, \mathbf{z}) = -\infty, \quad \overline{\pi}(t, \mathbf{z}) = \min_{j \in I \cup \{z_0\} \setminus M(\mathbf{z})} \left\{ -\frac{C(t, \mathbf{z})}{C(t, \mathbf{z}^j) - C(t, \mathbf{z})} \right\} \ .$$

(iii) If $C(t, \mathbf{z}^i) > C(t, \mathbf{z})$ for $i \in \widetilde{M}$, and $C(t, \mathbf{z}^j) < C(t, \mathbf{z})$ for $j \in \widetilde{M}'$,

$$\underline{\pi}(t, \mathbf{z}) = \max_{i \in \widetilde{M}} \left\{ -\frac{C(t, \mathbf{z})}{C(t, \mathbf{z}^i) - C(t, \mathbf{z})} \right\}, \quad \overline{\pi}(t, \mathbf{z}) = \min_{j \in \widetilde{M}'} \left\{ -\frac{C(t, \mathbf{z})}{C(t, \mathbf{z}^j) - C(t, \mathbf{z})} \right\} \ ,$$

where $\widetilde{M} \neq \emptyset$, $\widetilde{M}' \neq \emptyset$, $\widetilde{M} \cap \widetilde{M}' = \emptyset$, and $\widetilde{M} \cup \widetilde{M}' = I \cup \{z_0\} \setminus M(\mathbf{z})$.

Next, we introduce the investor's utility maximization problem.

3.2 Utility Maximization

Given a state $(t, w, \mathbf{z}) \in [0, T) \times \mathbb{R}_+ \times \mathcal{S}$ and a $(t, w, \mathbf{z})$-admissible feedback investment strategy $\pi \in \Pi(t, \mathbf{z})$, we assume that the investor's objective is the following expected utility of his terminal wealth:

$$J^T(t, w, \mathbf{z};\, \pi) = \mathbb{E}^{\mathbb{P}}[U(W^\pi(T)) \mid W^\pi(t-) = w,\, \mathbf{H}(t-) = \mathbf{z}] \ ,$$

where $\mathbb{E}^{\mathbb{P}}$ is the expectation with respect to the physical measure $\mathbb{P}$, the utility function $U(\,\cdot\,)$ is given by

$$U(w) = \frac{w^\gamma}{\gamma} \ ,$$

with $\gamma \in (0, 1)$ the risk-aversion parameter. Here W^π satisfies the dynamics (19) with $W^\pi(t-) = w$, and $\mathbf{H}$ follows a continuous-time Markov chain on $\mathcal{S}$ with a transition rate $\mathbf{1}_{\{H_i(t)=0\}} h_i^{\mathbb{P}}(\mathbf{H}(t))$ and an initial value $\mathbf{H}(t-) = \mathbf{z}$, where $h_i^{\mathbb{P}}(\,\cdot\,)$ is defined in (15) for $i \in I \cup \{0\}$.

For any $(t, w, \mathbf{z}) \in [0, T] \times \mathbb{R}_+ \times \mathcal{S}$, the investor's goal is to maximize the objective functional $J^T(t, w, \mathbf{z};\, \pi)$ across the class of the $(t, w, \mathbf{z})$-admissible feedback investment strategy π subject to the constraint $\Pi(t, \mathbf{z})$ given in Proposition 5. Hence, the investor's portfolio selection problem can be written as the following stochastic control problem:

$$v(t, w, \mathbf{z}) = \sup_{\pi \in \Pi(t, \mathbf{z})} J^T(t, w, \mathbf{z};\, \pi) \ , \tag{22}$$

subject to the dynamics (19), and the default intensity (15).

In the next section, we turn to a theoretical analysis to the utility maximization problem (22).

4 Theoretical Analysis

To solve the optimization problem (22), we shall analyze an associated Hamilton-Jacobi-Bellman (HJB) equation, which can be formally derived by applying the dynamic programming principle. The optimal investment strategies are then studied by assuming a smooth solution to the HJB equation exists. Finally, after proving the existence and uniqueness of a classic solution to the HJB equation, we establish a verification theorem for the original optimization problem (22).

4.1 HJB Equation

By the dynamic programming principle (e.g., Pham (2009)), the value function $v(t, w, \mathbf{z})$ defined in (22) is expected to solve an HJB equation coupled with suitable terminal and/or boundary conditions.

First, consider the region where the CLN is still alive, i.e., $(t, w, \mathbf{z}) \in [0, T) \times \mathbb{R}_+ \times \mathcal{Z}_k$, where the set $\mathcal{Z}_k$ is defined in (4). In this region, the value function satisfies the following equation:

$$\sup_{\pi \in \Pi(t,\mathbf{z})} \left(\frac{\partial}{\partial t} + \mathcal{L} \right) v\left(t, w, \mathbf{z}\right) = 0 \quad \text{ for } (t, w, \mathbf{z}) \in [0, T) \times \mathbb{R}_+ \times \mathcal{Z}_k \ . \tag{23}$$

Here $\Pi(t, \mathbf{z})$ is the constraint of all $(t, w, \mathbf{z})$-admissible investment strategies specified in Proposition 5. The operator $\mathcal{L} = \mathcal{L}_w + \mathcal{L}_J$, where

$$\mathcal{L}_w v(t, w, \mathbf{z}) = \Big(r - \pi \sum_{j \in I \cup \{z_0\} \setminus M(\mathbf{z})} \Delta^M(t, \mathbf{z}, \mathbf{z}^j) h_j(\mathbf{z}) \Big) w \frac{\partial}{\partial w} v(t, w, \mathbf{z}) \ , \tag{24}$$

$$\begin{aligned} \mathcal{L}_J v(t, w, \mathbf{z}) = & \sum_{j \in I \cup \{z_0\} \setminus M(\mathbf{z})} \Big(v\left(t, w + w\pi\Delta^M(t, \mathbf{z}, \mathbf{z}^j), \mathbf{z}^j\right) - v(t, w, \mathbf{z}) \Big) \\ & \times (1 + \lambda_j(\mathbf{z})) h_j(\mathbf{z}) \ . \end{aligned} \tag{25}$$

Here $\Delta^M(t, \mathbf{z}, \mathbf{z}^j)$ is defined in (20), and $h_j(\mathbf{z})$ is given by (3).

Second, consider the region where the CLN has been already terminated, i.e., $(t, w, \mathbf{z}) \in [0, T) \times \mathbb{R}_+ \times \mathcal{S} \setminus \mathcal{Z}_k$. Since $\Delta^M(t, \mathbf{z}, \mathbf{z}^j) = 0$ when $|M(\mathbf{z})| \geq k$, we have $dW(t) = r\, W(t)\, dt$ as pointed out in Remark 3.1. Consequently, the following condition must hold:

$$v(t, w, \mathbf{z}) = e^{\gamma r(T-t)} U(w) \quad \text{ for } (t, w, \mathbf{z}) \in [0, T) \times \mathbb{R}_+ \times \mathcal{S} \setminus \mathcal{Z}_k \ , \tag{26}$$

which is the Merton's solution with only risk-free asset.

Finally, at the terminal time of the investment T, we immediately obtain the following condition:

$$v\left(T, w, \mathbf{z}\right) = U(w) \quad \text{ for } (w, \mathbf{z}) \in \mathbb{R}_+ \times \mathcal{S} \ . \tag{27}$$

With the help of the HJB equation, we now consider the investment strategy.

4.2 Optimal Investment Strategy

We assume that the HJB equation (23) has a sufficiently smooth solution $v(t, w, \mathbf{z})$ with the following separation property in w:

$$v(t, w, \mathbf{z}) = w^\gamma V(t, \mathbf{z}) \ , \tag{28}$$

where $V(t, \mathbf{z})$ is a sufficiently smooth function for $(t, \mathbf{z}) \in [0, T] \times \mathcal{S}$. As in Merton (1969), this separation not only reduces our problem from three dimensions of $(t, w, \mathbf{z})$ to two dimensions of $(t, \mathbf{z})$, but also implies that the value function is concave in w, which is critical for analyzing the optimal strategy in terms of the proportion of wealth invested in the CLN denoted by $\pi^*(t) = \pi^*(t, w, \mathbf{z})$. We will continue our subsequent analysis with this conjecture, and later will verify it.

First of all, since the investor will not hold the CLN when it has been already terminated, the optimal strategy must be zero, i.e., $\pi^*(t) = 0$ for $(t, w, \mathbf{z}) \in [0, T) \times \mathbb{R}_+ \times \mathcal{S} \setminus \mathcal{Z}_k$. Hence, it is sufficient to study the region where the CLN is still alive, i.e., $(t, w, \mathbf{z}) \in [0, T) \times \mathbb{R}_+ \times \mathcal{Z}_k$.

Note that $v(t, w, \mathbf{z})$ given in (28) is concave in w, so $\pi^*(t)$ must satisfy the following first order condition: For $(t, w, \mathbf{z}) \in [0, T) \times \mathbb{R}_+ \times \mathcal{Z}_k$,

$$
\begin{aligned}
0 = \sum_{j \in I \cup \{z_0\} \setminus M(\mathbf{z})} & \frac{\partial}{\partial w} v\Big(t, w[1 + \pi^* \Delta^M(t, \mathbf{z}, \mathbf{z}^j)], \mathbf{z}^j\Big) \Delta^M(t, \mathbf{z}, \mathbf{z}^j)(1 + \lambda_j(\mathbf{z})) h_j(\mathbf{z}) \\
& - \frac{\partial}{\partial w} v(t, w, \mathbf{z}) \sum_{j \in I \cup \{z_0\} \setminus M(\mathbf{z})} \Delta^M(t, \mathbf{z}, \mathbf{z}^j) h_j(\mathbf{z}) \ , \qquad (29)
\end{aligned}
$$

where $\Delta^M(t, \mathbf{z}, \mathbf{z}^j)$ is defined in (20), and $h_j(\mathbf{z})$ is defined in (15). To interpret the first order condition (29), we rewrite it as

$$
\sum_{j \in I \cup \{z_0\} \setminus M(\mathbf{z})} \omega_j (1 + \lambda_j(\mathbf{z})) \frac{\partial}{\partial w} v\Big(t, w[1 + \pi^* \Delta^M(t, \mathbf{z}, \mathbf{z}^j)], \mathbf{z}^j\Big) = \frac{\partial}{\partial w} v(t, w, \mathbf{z}) \ , \qquad (30)
$$

where the "weight" ω_j is given by

$$
\omega_j = \frac{\Delta^M(t, \mathbf{z}, \mathbf{z}^j) h_j(\mathbf{z})}{\sum_{j \in I \cup \{z_0\} \setminus M(\mathbf{z})} \Delta^M(t, \mathbf{z}, \mathbf{z}^j) h_j(\mathbf{z})} \ .
$$

Which event (default or shock realization) will occur is *a priori* unknown, so (30) implies that the investor chooses the optimal π^* to make the current marginal value of wealth (i.e., $\frac{\partial}{\partial w} v(t, w, \mathbf{z})$) equal the weighted average of the risk-adjusted marginal value of wealth conditional on a default or a shock realization. Here the weight ω_j is determined jointly by the jump size $\Delta^M(t, \mathbf{z}, \mathbf{z}^j)$ and the jump intensity $h_j(\mathbf{z})$. The marginal value conditional on a default or a shock realization is adjusted by the corresponding risk compensation characterized by $(1 + \lambda_j(\mathbf{z}))$. In other words, the first order condition (30) suggests that the investor needs to balance all risk factors in the reference pool and from the external shock simultaneously. Unlike the case where each asset associates with a single specific risk factor, the CLN contains all these risk factors. As a consequence, these risk factors are non-separable, which makes the optimal investment strategies more involved.

Using the homogeneity property, we can simplify the first order condition (29) as the following one, which is independent of the wealth level w:

$$0 = \sum_{j\in I\cup\{z_0\}\setminus M(\mathbf{z})} V(t,\mathbf{z}^j)\left(1+\pi^*(t)\,\Delta^M(t,\mathbf{z},\mathbf{z}^j)\right)^{\gamma-1}\Delta^M(t,\mathbf{z},\mathbf{z}^j)(1+\lambda_j(\mathbf{z}))h_j(\mathbf{z})$$
$$-V(t,\mathbf{z})\sum_{j\in I\cup\{z_0\}\setminus M(\mathbf{z})}\Delta^M(t,\mathbf{z},\mathbf{z}^j)h_j(\mathbf{z})\ . \tag{31}$$

Since the first order condition (31) is highly nonlinear, $\pi^*(t,\mathbf{z})$ generally cannot be derived in a closed form. To examine the existence and uniqueness of the optimal strategy $\pi^*(t)$, define a strictly positive vector

$$\mathbf{V}=\left(V,V^{j_1},\cdots,V^{j_{|I\cup\{z_0\}\setminus M(\mathbf{z})|}}\right),\quad j_i\in I\cup\{z_0\}\setminus M(\mathbf{z})\ , \tag{32}$$

and a function

$$g(t,\mathbf{z},\mathbf{V},\pi)=\sum_{j\in I\cup\{z_0\}\setminus M(\mathbf{z})} V^j\left(1+\pi\Delta^M(t,\mathbf{z},\mathbf{z}^j)\right)^{\gamma-1}\Delta^M(t,\mathbf{z},\mathbf{z}^j)(1-\lambda_j(\mathbf{z}))h_j(\mathbf{z})$$
$$-V\sum_{j\in I\cup\{z_0\}\setminus M(\mathbf{z})}\Delta^M(t,\mathbf{z},\mathbf{z}^j)h_j(\mathbf{z})\ , \tag{33}$$

for $(t,\mathbf{z})\in[0,T)\times\mathcal{Z}_k$, $\mathbf{V}$ is given in (32), and $\pi\in\Pi(t,\mathbf{z})$, where $\Pi(t,\mathbf{z})$ is given in Proposition 5.

Proposition 6. *For all $(t,\mathbf{z})\in[0,T)\times\mathcal{Z}_k$, let the vector $\mathbf{V}$, the function $g(t,\mathbf{z},\mathbf{V},\pi)$ and the set $\Pi(t,\mathbf{z})$ be given by (32), (33), and Proposition 5, respectively. Then, there exists a unique $\pi^*\in\Pi(t,\mathbf{z})$ such that*

$$g(t,\mathbf{z},\mathbf{V},\pi^*)=0\ . \tag{34}$$

In addition, π^* is continuously differentiable in $(t,\mathbf{z},\mathbf{V})$.

Proof. Note first that $g(t,\mathbf{z},\mathbf{V},\pi)$ is continuous and decreasing with respect to π, and $\mathbf{V}>0$. According to the signs of Δ^M, we next divide the proof into several cases:

Case 1. $\frac{C(t,\mathbf{z}^j)}{C(t,\mathbf{z})}-1>0$, $j\in I\cup\{z_0\}\setminus M(\mathbf{z})$. It is easy to check that

$$\begin{cases}\lim\limits_{\pi\downarrow\underline{\pi}} g(t,\mathbf{z},\mathbf{V},\pi)=+\infty\ ,\\ \lim\limits_{\pi\uparrow\overline{\pi}} g(t,\mathbf{z},\mathbf{V},\pi)=V(t,\mathbf{z})\left[\sum\limits_{j\in I\cup\{z_0\}\setminus M(\mathbf{z})}\left(1-\dfrac{C(t,\mathbf{z}^j)}{C(t,\mathbf{z})}\right)h_j(\mathbf{z})\right]<0\ ,\end{cases}$$

where $\underline{\pi}$ and $\overline{\pi}$ are given in Proposition 5. Since $g(t, \mathbf{z}, \mathbf{V}, \pi)$ is continuous in $\pi \in (\underline{\pi}, \overline{\pi})$, the Intermediate Value Theorem implies that there is a unique solution $\pi^* \in (\underline{\pi}, \overline{\pi})$ to the equation (34).

Case 2. $\frac{C(t,\mathbf{z}^j)}{C(t,\mathbf{z})} - 1 < 0$, $j \in I \cup \{z_0\} \setminus M(\mathbf{z})$. It is easy to check that

$$\begin{cases} \lim\limits_{\pi \downarrow \underline{\pi}} g(t, \mathbf{z}, \mathbf{V}, \pi) = V(t, \mathbf{z}) \Bigg[\sum\limits_{j \in I \cup \{z_0\} \setminus M(\mathbf{z})} \Big(1 - \dfrac{C(t, \mathbf{z}^j)}{C(t, \mathbf{z})}\Big) h_j(\mathbf{z}) \Bigg] > 0 \ , \\ \lim\limits_{\pi \uparrow \overline{\pi}} g(t, \mathbf{z}, \mathbf{V}, \pi) = -\infty \ . \end{cases}$$

Hence, there exists a unique solution $\pi^* \in (\underline{\pi}, \overline{\pi})$ to the equation (34) in the desired domain of π.

Case 3. $\frac{C(t,\mathbf{z}^i)}{C(t,\mathbf{z})} - 1 > 0$, $i \in M$, $\frac{C(t,\mathbf{z}^j)}{C(t,\mathbf{z})} - 1 < 0$, $j \in M'$, $M \neq \emptyset$, $M' \neq \emptyset$, $M \cup M' = I \cup \{z_0\} \setminus M(\mathbf{z})$ and $M \cap M' = \emptyset$. Then, we have

$$\lim_{\pi \downarrow \underline{\pi}} g(t, \mathbf{z}, \mathbf{V}, \pi) = +\infty, \quad \lim_{\pi \uparrow \overline{\pi}} g(t, \mathbf{z}, \mathbf{V}, \pi) = -\infty \ ,$$

Consequently, there exists a unique solution $\pi^* \in (\underline{\pi}, \overline{\pi})$ to the equation $g(t, \mathbf{z}, \mathbf{V}, \pi) = 0$.

Case 4. $\frac{C(t,\mathbf{z}^i)}{C(t,\mathbf{z})} - 1 = 0$, $\frac{C(t,\mathbf{z}^j)}{C(t,\mathbf{z})} - 1 \neq 0$, $i \in K$, $j \in I \cup \{z_0\} \setminus M(\mathbf{z}) \setminus K$, where $K \subset I \cup \{z_0\} \setminus M(\mathbf{z})$. Using a similar argument, we can also get the unique solution of equation (34).

Finally, the claim that $\pi^*(t, \mathbf{z})$ is continuously differentiable in $(t, \mathbf{z}, \mathbf{V})$ follows immediately from the Implicit Function Theorem. □

Note that if $V(t, \mathbf{z}) > 0$ for all $(t, \mathbf{z}) \in [0, T) \times \mathcal{Z}_k$, we can set $V = V(t, \mathbf{z})$, and $V^{j_i} = V(t, \mathbf{z}^{j_i})$ for $j_i \in I \cup \{z_0\} \setminus M(\mathbf{z})$. Applying Propositiowen 6, the optimal investment strategy $\pi^*(t)$ can be derived uniquely from the first order condition (31).

4.3 ODE System for the Reduced Value Function V

As mentioned before, using the separation property in (28), we can reduce the three-dimensional problem for the original value function $v(t, w, \mathbf{z})$ to the two-dimensional problem for the reduced value function $V(t, \mathbf{z})$.

First, in the region where the CLN is alive, i.e., $(t, \mathbf{z}) \in [0, T) \times \mathcal{Z}_k$, $V(t, \mathbf{z})$ solves the following ODEs:

$$\begin{aligned} 0 = \max_{\pi \in \Pi(t,\mathbf{z})} & \frac{dV(t, \mathbf{z})}{dt} + \gamma V(t, \mathbf{z}) \Big\{ r - \pi \sum_{j \in I \cup \{z_0\} \setminus M(\mathbf{z})} \Delta^M(t, \mathbf{z}, \mathbf{z}^j) h_j(\mathbf{z}) \Big\} \\ & + \sum_{j \in I \cup \{z_0\} \setminus M(\mathbf{z})} \left\{ \big(1 + \pi \, \Delta^M(t, \mathbf{z}, \mathbf{z}^j)\big)^\gamma V(t, \mathbf{z}^j) - V(t, \mathbf{z}) \right\} (1 + \lambda_j(\mathbf{z})) h_j(\mathbf{z}) \ . \end{aligned} \tag{35}$$

Second, in the region where the CLN has been already terminated, i.e., $(t, \mathbf{z}) \in [0, T) \times \mathcal{S} \setminus \mathcal{Z}_k$, the condition (26) implies that

$$V(t, \mathbf{z}) = \frac{1}{\gamma} e^{\gamma r(T-t)} \ . \tag{36}$$

Finally, the terminal condition (27) is equivalent to that

$$V(T, \mathbf{z}) = \frac{1}{\gamma} \quad \text{for } z \in \mathcal{S} \ . \tag{37}$$

The following proposition states that the above ODE system is well-posed.

Proposition 7 (Existence and Uniqueness)**.** *The nonlinear ODE system of* (35)*,* (36)*, and* (37) *has a unique positive solution* $V(t, \mathbf{z})$*, which is continuously differentiable in* $t \in [0, T]$ *for all* $\mathbf{z} \in \mathcal{Z}_k$*.*

Proof. We prove the proposition by first constructing a sequence of functions $\{\varphi_n\}_{n=0}^{\infty}$, and then showing that the constructed sequence converges to the solution of the nonlinear ODE system. We divide the construction into several steps below.

Step 1. Fix $\mathbf{z} \in \mathcal{S}$, and define $\varphi_0(t, \mathbf{z}) = \frac{1}{\gamma} e^{\gamma r(T-t)}$ for $t \in [0, T]$. Let

$$\mathbf{V} = \left(\varphi_0, \varphi_0(t, \mathbf{z}^{j_1}), \cdots, \varphi_0(t, \mathbf{z}^{j_{|I \cup \{z_0\} \setminus M(\mathbf{z})|}})\right) \ .$$

Applying Proposition 6, there is a unique $\pi_0 \in \Pi(t, \mathbf{z})$ such that $g(t, \mathbf{z}, \mathbf{V}, \pi_0) = 0$, where $g(\,\cdot\,)$ is defined in (33). In addition, the continuity of φ_0 implies that π_0 is also continuously differentiable in $t \in [0, T]$, and is uniformly bounded.

Step 2. Consider the following *linear* ODE system of (26), (27), and

$$\begin{aligned} 0 = {} & \frac{d\varphi(t, \mathbf{z})}{dt} + \gamma \varphi(t, \mathbf{z}) \Big\{ r - \pi_0 \sum_{j \in I \cup \{z_0\} \setminus M(\mathbf{z})} \Delta^M(t, \mathbf{z}, \mathbf{z}^j) h_j(\mathbf{z}) \Big\} \\ & + \sum_{j \in I \cup \{z_0\} \setminus M(\mathbf{z})} \left\{ \left(1 + \pi_0 \, \Delta^M(t, \mathbf{z}, \mathbf{z}^j)\right)^\gamma \varphi(t, \mathbf{z}^j) - \varphi(t, \mathbf{z}) \right\} (1 + \lambda_j(\mathbf{z})) h_j(\mathbf{z}) \ , \end{aligned} \tag{38}$$

in $[0, T]$. A standard ODE theory shows that the above linear ODE system admits a classical solution, which is denoted by $\varphi_1(t, \mathbf{z})$ for $t \in [0, T]$. Moreover, we have

$$\varphi_1(t, \mathbf{z}) > 0, \quad ||\varphi_1|| = \max_{t \in [0,T]} \left\{ |\varphi_1(t, \mathbf{z})| + \left| \frac{d\varphi_1(t, \mathbf{z})}{dt} \right| + \left| \frac{d^2 \varphi_1(t, \mathbf{z})}{dt^2} \right| \right\} < C_T \ ,$$

where C_T is a positive constant depending only on T. To see this, we rewrite the ODE (38) in the following way:

$$0 = \frac{d\varphi(t, \mathbf{z})}{dt} + \varphi(t, \mathbf{z}) \Big\{ \gamma r - \sum_{j \in I \cup \{z_0\} \setminus M(\mathbf{z})} \Big[\gamma \pi_0 \Delta^M(t, \mathbf{z}, \mathbf{z}^j) + (1 + \lambda_j(\mathbf{z})) \Big] h_j(\mathbf{z}) \Big\}$$

$$+ \sum_{j \in I \cup \{z_0\} \setminus M(\mathbf{z})} \left(1 + \pi_0 \, \Delta^M(t, \mathbf{z}, \mathbf{z}^j)\right)^\gamma \varphi(t, \mathbf{z}^j)(1 + \lambda_j(\mathbf{z}))h_j(\mathbf{z}) \ .$$

Then, we can follow the arguments in the proof of Proposition 1 by first considering the case of $|M(\mathbf{z})| = k-1$ and using the condition (36) in the early termination region to show that the last term in the above ODE, i.e., $\sum \left(1 + \pi_0 \, \Delta^M(t, \mathbf{z}, \mathbf{z}^j)\right)^\gamma \varphi(t, \mathbf{z}^j)(1+\lambda_j(\mathbf{z}))h_j(\mathbf{z})$, is positive, continuous, and uniformly bounded. By the theory of linear ODE, $\varphi_1(t, \mathbf{z})$ satisfies the above stated properties when $|M(\mathbf{z})| = k-1$. For the case of $|M(\mathbf{z})| < k-1$, a similar argument leads to the required results.

Step 3. As in *Step 1*, update a new strategy π_1 by again applying Proposition 6 with replacing φ_0 by φ_1. Next, solve the linear ODE system of (36), (37), and (38) with replacing π_0 by π_1, and denote its solution by φ_2.

Repeat the above procedure, we obtain a sequence of functions $\{\varphi_n(t, \mathbf{z})\}_{n=0}^\infty$ satisfying that φ_n is continuously differentiable in $t \in [0, T]$, and

$$\varphi_n(t, \mathbf{z}) > 0, \quad ||\varphi_n|| < C_T \ ,$$

where C_T is a positive constant depending only on T. Meanwhile, we also have a sequence of investment strategies $\{\pi_n(t)\}_{n=0}^\infty$ which are continuous and uniformly bounded.

Next, we prove the convergence of the sequence to the solution of the nonlinear ODE system. By Arzela-Ascoli lemma, there exists a continuously differentiable function $\varphi_*(t, \mathbf{z}) = \lim_{n\to\infty} \varphi_n(t, \mathbf{z})$ and a continuous strategy $\pi_*(t) = \lim_{n\to\infty} \pi_n(t)$ solving the linear ODE system of (36), (37), and (38) with π_0 being replaced by π_*. Moreover, π_* also solves the equation $g(t, \mathbf{z}, \mathbf{V}_*, \pi_*) = 0$, where $V_* = \left(\varphi_*, \varphi_*(t, \mathbf{z}^{j_1}), \cdots, \varphi_*(t, \mathbf{z}^{j_{|I \cup \{z_0\} \setminus M(\mathbf{z})|}})\right)$. This in turn implies that φ_* solves the nonlinear ODE system of (35), (36), and (37).

Finally, we turn to the uniqueness. For $(t, \mathbf{z}) \in [0, T] \times \mathscr{Z}_k$, and a continuously differentiable function $\varphi(t, \mathbf{z})$, define

$$\begin{aligned} F(\varphi) = {} & \gamma\varphi(t, \mathbf{z})\Big\{r - \pi(\varphi) \sum_{j \in I \cup \{z_0\} \setminus M(\mathbf{z})} \Delta^M(t, \mathbf{z}, \mathbf{z}^j) h_j(\mathbf{z})\Big\} \\ & + \sum_{j \in I \cup \{z_0\} \setminus M(\mathbf{z})} \left\{\left(1 + \pi(\varphi) \, \Delta^M(t, \mathbf{z}, \mathbf{z}^j)\right)^\gamma \varphi(t, \mathbf{z}^j) - \varphi(t, \mathbf{z})\right\} (1 + \lambda_j(\mathbf{z}))h_j(\mathbf{z}) \ , \end{aligned}$$

where $\pi(\varphi)$ is the unique solution derived in Proposition 6 with $\mathbf{V} = \left(\varphi, \varphi(t, \mathbf{z}^{j_1}), \cdots, \varphi(t, \mathbf{z}^{j_{|I \cup \{z_0\} \setminus M(\mathbf{z})|}})\right)$. Then, Proposition 6 shows that $\pi(\varphi)$ is continuously differentiable in φ. This in turn implies that $F(\varphi)$ is also continuously differentiable and thereby locally Lipschitz. So, the uniqueness follows immediately from the uniqueness of the nonlinear ode $\frac{d}{dt}\varphi = F(\varphi)$. $\square$

4.4 Verification Theorem

Having established the existence and uniqueness of a sufficiently smooth solution to the HJB equation (23) together with the conditions given in (26) and (27), we now show that this smooth function coincides with the value function v defined in (22).

Recall that $\mathcal{L} = \mathcal{L}_w + \mathcal{L}_J$, where $\mathcal{L}_w$ and $\mathcal{L}_J$ are defined in (24) and (25), respectively. Let $\varphi(t, w, \mathbf{z})$ be a C^1 in t and w. Also, let $\pi(t) \in \Pi(t, \mathbf{z})$. Then for $t < u$,

$$\begin{aligned}\varphi(u, W^\pi(u), \mathbf{H}(u)) = \varphi(t, W^\pi(t-), \mathbf{H}(t-)) + \int_t^u \left(\frac{\partial}{\partial s} + \mathcal{L}\right) \varphi(s, W^\pi(s), \mathbf{H}(s)) ds \\ + \mathcal{M}^\pi_\varphi(u) - \mathcal{M}^\pi_\varphi(t),\end{aligned}$$

where W^π is the wealth process under the control π, and $\mathcal{M}^\pi(t)$ is a $\mathbb{P}$-(local) martingale defined by

$$\mathcal{M}^\pi_\varphi(t) = \sum_{j \in I \cup \{z_0\} \setminus M(\mathbf{H}(t-))} \int_0^t \left[\varphi\Big(s, \widetilde{W}^\pi_j(s-), \mathbf{H}^j(s-)\Big) - \varphi\Big(s, W^\pi(s-), \mathbf{H}(s-)\Big)\right] d\xi^{\mathbb{P}}_j(s)$$

for $t \geq 0$, and $\widetilde{W}^\pi_j(s-) = W^\pi(s-)[1 + \pi(s-)\Delta^M(s, \mathbf{H}(s-), \mathbf{H}^j(s-))]$.

Theorem 1 (Verification)**.** *Let $V(t, \mathbf{z})$ be the unique positive solution to the ODE system of* (35)*,* (36)*, and* (37) *in $[0, T] \times \mathcal{S}$. For all $(t, \mathbf{z}) \in [0, T] \times \mathcal{S}$, define the optimal strategies $\pi^*(t)$ as follows:*

$$\pi^*(t) = \begin{cases} \text{satisfies the first-order condition (31)}, & \text{if } |M(\mathbf{z})| \leq k - 1 \ , \\ 0, & \text{if } |M(\mathbf{z})| \geq k \ . \end{cases} \tag{39}$$

Then, for all $(t, w, \mathbf{z}) \in [0, T] \times \mathbb{R}_+ \times \mathcal{S}$, the value function $v(t, w, \mathbf{z})$ defined in (22) *has the following form*

$$v(t, w, \mathbf{z}) = w^\gamma V(t, \mathbf{z}) \ .$$

In addition, the optimal investment strategy in terms of proportional wealth invested in the CLN is given by $\pi^*(t)$ defined in (39)*.*

Proof. First, we prove that the solution of the HJB equation is no bigger than the value function. Second, we prove that the solution of the HJB equation is consistent with the value function when taking the limits. Divide the construction into several steps below.

Step 1. Let $\varphi(t, w, \mathbf{z}) = w^\gamma V(t, \mathbf{z})$ for $(t, w, \mathbf{z}) \in [0, T] \times \mathbb{R}_+ \times \mathcal{S}$. For any admissible feedback control $\pi(t) \in \Pi(t, \mathbf{z})$, define the process

$$Y^\pi(u) = \varphi\big(u, W^\pi(u), \mathbf{H}(u)\big) \quad , u \geq t \ ,$$

where $W^\pi(u)$ is the wealth process under the control π with initial state $W^\pi(t-) = w$.

According to Ito's formula, $Y^\pi(u)$ satisfies

$$Y^\pi(u) = Y^\pi(t-) + \int_t^u F\big(\pi(s), s, W^\pi(s), \mathbf{H}(s)\big)ds + \mathcal{M}^\pi_\varphi(u) - \mathcal{M}^\pi_\varphi(t),$$

where the function $F(\pi, t, w, \mathbf{z})$ is given by

$$\begin{aligned} F(\pi, t, w, \mathbf{z}) = {} & w^\gamma \frac{\partial V(t,\mathbf{z})}{\partial t} + \gamma w^\gamma \Big[r - \pi \sum_{j\in I\cup\{z_0\}\setminus M(\mathbf{z})} \Delta^M(t,\mathbf{z},\mathbf{z}^j) h_j(\mathbf{z})\Big] V(t,\mathbf{z}) \\ & + w^\gamma \sum_{j\in I\cup\{z_0\}\setminus M(\mathbf{z})} \Big[\Big(1+\pi\Delta^M(t,\mathbf{z},\mathbf{z}^j)\Big)^\gamma V(t,\mathbf{z}^j) - V(t,\mathbf{z})\Big] h_j^{\mathbb{P}}(\mathbf{z}). \end{aligned}$$

Note that

$$\frac{\partial^2 F}{\partial \pi^2} = \gamma(\gamma-1) w^\gamma \sum_{j\in I\cup\{z_0\}\setminus M(\mathbf{z})} \Big[1+\pi\Delta^M(t,\mathbf{z},\mathbf{z}^j)\Big]^{\gamma-2} \Big[\Delta^M(t,\mathbf{z},\mathbf{z}^j)\Big]^2 V(t,\mathbf{z}^j) h_j^{\mathbb{P}}(\mathbf{z}) \le 0 \ ,$$

which implies that

$$F(\pi, t, w, \mathbf{z}) \le F(\pi^*, t, w, \mathbf{z}) = 0,$$

where π^* is given by (39), and the last equality follows from Proposition 6. Consequently,

$$\mathbb{E}^{\mathbb{P}}_t[Y^\pi(u)] \le w^\gamma V(t,\mathbf{z}) + \mathbb{E}^{\mathbb{P}}_t[\mathcal{M}^\pi_\varphi(u) - \mathcal{M}^\pi_\varphi(t)] \ , \tag{40}$$

with equality when $\pi = \pi^*$, where $\mathbb{E}^{\mathbb{P}}_t[\cdot] = \mathbb{E}^{\mathbb{P}}[\cdot|\mathcal{G}_t]$.

Taking $u = T\wedge\tau_{a,b}$ in (C–74), we have $\mathbb{E}^{\mathbb{P}}_t[\mathcal{M}^\pi_\varphi(u) - \mathcal{M}^\pi_\varphi(t)] = 0$, where $\tau_{a,b} = \inf\{u \ge t; W^\pi(u) \ge b^{-1}$ or $W^\pi(u) \le a\}$ and $0 < a < w < b^{-1}$. Hence

$$\mathbb{E}^{\mathbb{P}}_t[Y^\pi(T\wedge\tau_{a,b})] \le w^\gamma V(t,\mathbf{z}) \ ,$$

with the equality when $\pi = \pi^*$. Since $Y^\pi(T\wedge\tau_{a,b})$ is positive, using Fatou lemma and the terminal condition (37) yields that

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}_t[U(W^\pi(T))] & \le \lim_{a,b\to 0} \mathbb{E}^{\mathbb{P}}_t\Big[\Big(W^\pi(T\wedge\tau_{a,b})\Big)^\gamma V(T\wedge\tau_{a,b}, \mathbf{H}(T\wedge\tau_{a,b}))\Big] \\ & = \lim_{a,b\to 0} \mathbb{E}^{\mathbb{P}}_t[Y^\pi(T\wedge\tau_{a,b})] \le w^\gamma V(t,\mathbf{z}) \quad \text{for all } \pi \in \Pi(t,\mathbf{z}) \ . \end{aligned}$$

Consequently, from the definition (28) we have

$$v(t,w,\mathbf{z}) = \sup_{\pi\in\Pi(t,\mathbf{z})} \mathbb{E}^{\mathbb{P}}_t[U(W^\pi(T))] \le w^\gamma V(t,\mathbf{z}) \ .$$

Step 2. It remains to show that $v(t,w,\mathbf{z}) = w^\gamma V(t,\mathbf{z})$ under the optimal investment strategy π^*. To this, it suffices to show that

$$\lim_{a,b\to 0} \mathbb{E}^{\mathbb{P}}_t\Big[\Big(W^{\pi^*}(T\wedge\tau_{a,b})\Big)^\gamma V(T\wedge\tau_{a,b}, \mathbf{H}(T\wedge\tau_{a,b}))\Big] = \mathbb{E}^{\mathbb{P}}_t[U(W^{\pi^*}(T))] \ . \tag{41}$$

Note that from the continuous differentiability of the function $V(t,\mathbf{z})$ in $t \in [0,T]$, there exist $C_1 > 0$ and $C_2 > 0$ such that

$$\lim_{a,b\to 0} \mathbb{E}_t^{\mathbb{P}}\Big[\Big|\Big(W^{\pi}(T\wedge\tau_{a,b})\Big)^{\gamma} V(T\wedge\tau_{a,b},\mathbf{H}(T\wedge\tau_{a,b}))\Big|^2\Big] \leq C_1 + C_2\,\mathbb{E}_t^{\mathbb{P}}\Big[\Big|W^{\pi}(T\wedge\tau_{a,b})\Big|^2\Big].$$

By the Corollary 7.1.5 in Chow and Teicher (2003), (D–76) holds if we can show that

$$\sup_{0<a<w<b^{-1}<+\infty} \mathbb{E}_t^{\mathbb{P}}\Big[\Big|W^{\pi}(T\wedge\tau_{a,b})\Big|^2\Big] < \infty\ . \tag{42}$$

To show (42), let's first define the functions respectively

$$\begin{aligned}
\alpha(t,w,\mathbf{z}) &= w\Big(r - \pi^* \sum_{j\in I\cup\{z_0\}\setminus M(\mathbf{z})} \Delta^M(t,\mathbf{z},\mathbf{z}^j)\lambda_j h_j(\mathbf{z})\Big)\ ,\\
\beta_j(t,w,\mathbf{z}) &= w\,\pi^*\,\Delta^M(t,\mathbf{z},\mathbf{z}^j),\quad j\in I\cup\{z_0\}\setminus M(\mathbf{z})\ .
\end{aligned}$$

Then, the $\mathbb{P}$-dynamics of the wealth process can be written as

$$dW^{\pi^*}(t) = \alpha(t,W^{\pi^*}(t),\mathbf{H}(t))dt + \sum_{j\in I\cup\{z_0\}\setminus M(\mathbf{z})} \beta_j(t,W^{\pi^*}(t),\mathbf{H}(t))d\xi_j^{\mathbb{P}}(t).$$

Because $\underline{\pi} < \pi^* < \overline{\pi}$, we consider the following two cases:
Case (i). $-\infty < \underline{\pi} < \overline{\pi} < +\infty$.

Let $D_{N,T}$ be a generic constant depending on N and T, which may be different for each inequality. For $0 \leq t < u \leq T$, the linear growth of $\alpha(t,w,\mathbf{z})$ in w implies that

$$\begin{aligned}
&\mathbb{E}_t^{\mathbb{P}}\Big[\sup_{t\leq u\leq T}\Big|\int_t^u \alpha(s,W^{\pi^*}(s)),\mathbf{H}(s))ds\Big|^2\Big]\\
\leq& D_{N,T}(T-t)\mathbb{E}_t^{\mathbb{P}}\Big[\int_t^T \Big(W^{\pi^*}(s)-w+w\Big)^2 ds\Big]\\
\leq& D_{N,T}\Big((T-t)w^2 + \mathbb{E}_t^{\mathbb{P}}\Big[\int_t^T \Big|W^{\pi^*}(s)-w\Big|^2 ds\Big]\Big),
\end{aligned}$$

where the last inequality follows from the Holder inequality. Similarly, the linear growth of $\beta_j(t,w,\mathbf{z})$ in w leads to that

$$\begin{aligned}
&\mathbb{E}_t^{\mathbb{P}}\Big[\sup_{t\leq u\leq T}\Big|\sum_{j\in I\cup\{z_0\}\setminus M(\mathbf{z})}\int_t^u \beta_j(s,W^{\pi^*}(s),\mathbf{H}(s))d\xi_j^{\mathbb{P}}(s)\Big|^2\Big]\\
\leq& D_{N,T}\sum_{j\in I\cup\{z_0\}\setminus M(\mathbf{z})}\mathbb{E}_t^{\mathbb{P}}\Big[\int_t^T \Big|\beta_j(s,W^{\pi^*}(s),\mathbf{H}(s))\Big|^2 dH_j(s)\Big]\\
=& D_{N,T}\sum_{j\in I\cup\{z_0\}\setminus M(\mathbf{z})}\mathbb{E}_t^{\mathbb{P}}\Big[\int_t^T \Big|\beta_j(s,W^{\pi^*}(s),\mathbf{H}(s))\Big|^2 h_j^{\mathbb{P}}(\mathbf{H}(s))ds\Big]\\
\leq& D_{N,T}\Big((T-t)w^2 + \mathbb{E}_t^{\mathbb{P}}\Big[\int_t^T \Big|W^{\pi^*}(s)-w\Big|^2 ds\Big]\Big),
\end{aligned}$$

where the last inequality is derived from the BDG inequality.

By the Gronwall inequality, the wealth process satisfies:

$$\mathbb{E}_t^{\mathbb{P}}\left[\sup_{t\leq u\leq T}|W^{\pi^*}(u)-w|^2\right]\leq D_{N,T}\,w^2+\bar{D}_{N,T}, \tag{43}$$

where $\bar{D}_{N,T}$ is a positive constant depending on N and T.

Case (ii). $\underline{\pi}=-\infty$ or $\overline{\pi}=+\infty$.

Since $V(t,\mathbf{z})$ is continuously differentiable in $t\in[0,T]$, Proposition 6 shows that $\pi^*=\pi^*(t,\mathbf{V}(t))$ is also continuous in $t\in[0,T]$, where the vector $\mathbf{V}(t)$ is defined by

$$\mathbf{V}(t)=\Big(V(t,\mathbf{z}),V(t,\mathbf{z}^{i_1}),\cdots,V(t,\mathbf{z}^{i_{|I\cup\{z_0\}\setminus M(\mathbf{z})|}})\Big).$$

So $\pi^*(t,\mathbf{z})$ is bounded. Then, we can also obtain the moment estimate (43) by using the similar argument in *Case (i)*.

Finally, from (43), we obtain

$$\sup_{0<a<w<b^{-1}<+\infty}\mathbb{E}_t^{\mathbb{P}}\left[|W^{\pi^*}(T\wedge\tau_{a,b})|^2\right]\leq 2w^2+2\mathbb{E}_t^{\mathbb{P}}\left[\sup_{t\leq u\leq T}\left|W^{\pi^*}(u)-w\right|^2\right]<+\infty.$$

This completes the proof. □

Before proceeding to a numerical analysis, we next focus on a special symmetric case where all reference entities are identical. In this case, the optimal investment strategy can be characterized more explicitly.

5 Identical Reference Entities

In this section, we focus on a special case where all reference entities have the same characteristics. To be more precise, for the external shock, we set

$$h_0(\mathbf{z})=a_0,\quad \alpha_{0,i}=\alpha_0\quad\text{for all } i\in I,$$

where $a_0>0$ and $\alpha_0>0$ are constants. For the reference entities, we assume that

$$a_i=\widehat{a},\quad \alpha_{ij}=\widehat{\alpha}\quad\text{for all } i,j\in I,$$

where $\widehat{a}>0$ and $\widehat{\alpha}>0$ are constants. In addition, the risk premium parameters of the external shock $\lambda_0(\mathbf{z})=\lambda_0\in(-1,\infty)$ and of the default $\lambda_i(\mathbf{z})=\widehat{\lambda}\in(-1,\infty)$, and the recovery rate $R_i=\widehat{R}\in[0,1]$ are the same for all reference entities.

Under this symmetric assumption, we can simplify the problem significantly. In particular, closed form solutions are available when there are no external contagion risks. To start, we first simplify our notations in the next subsection.

5.1 Notations

With the above symmetry, it is easy to check that the default intensity $h_i(\mathbf{z})$ can be rewritten as, for $i \in I \setminus M(\mathbf{z})$,

$$h_i(\mathbf{z}) = \begin{cases} \widehat{h}_{0,m} = \widehat{a}\,(\widehat{\alpha})^m & \text{if the external shock is not realized,} \\ \widehat{h}_{1,m} = \widehat{a}\,(\widehat{\alpha})^m \alpha_0 & \text{if the external shock is realized,} \end{cases}$$

which depends only on the default times $m = |M(\mathbf{z})|$ and the external shock realization. Put differently, the investor need only record the number of defaults in the reference pool but *not* the specific states of reference entities. Similarly, $C(t, \mathbf{z})$, the pre-termination value of the kth-to-default CLN, and $V(t, \mathbf{z})$, the reduced value function, also depend only on the default times $m = |M(\mathbf{z})|$ and the external shock realization. Hence, we use $\widehat{C}_{0,m}(t)$ and $\widehat{C}_{1,m}(t)$ to denote the values of CLN before and after the one-time external shock realization when there have been already m defaults, respectively. Similarly, $V_{0,m}(t)$ and $V_{1,m}(t)$ stands for the reduce value functions before and after the external shock realization when there have been already m defaults, respectively.

5.2 Values of CLN

From the results in Section 2 and using the symmetric property, it is easy to see that $\widehat{C}_{0,m}(t)$ and $\widehat{C}_{1,m}(t)$ solve respectively the following two ordinary differential equations:

$$\begin{cases} 0 = \frac{d}{dt}\widehat{C}_{0,m}(t) + \Big((N-m)\widehat{h}_{0,m} + r + a_0\Big)\widehat{C}_{0,m}(t) \\ \qquad\quad -(N-m)\widehat{h}_{0,m}\widehat{C}_{0,m+1}(t) - a_0\widehat{C}_{1,m}(t) - \kappa, \qquad t \in [0, \overline{T})\ , \\ \widehat{C}_{0,m}(\overline{T}) = 1, \quad 0 \le m < k \le N, \\ \widehat{C}_{0,k}(t) = \widehat{R}\ , \end{cases} \tag{44}$$

$$\begin{cases} 0 = \frac{d}{dt}\widehat{C}_{1,m}(t) + \Big((N-m)\widehat{h}_{1,m} + r\Big)\widehat{C}_{1,m}(t) \\ \qquad\quad -(N-m)\widehat{h}_{1,m}\widehat{C}_{1,m+1}(t) - \kappa, \qquad t \in [0, \overline{T})\ , \\ \widehat{C}_{1,m}(\overline{T}) = 1, \quad 0 \le m < k \le N, \\ \widehat{C}_{1,k}(t) = \widehat{R}\ . \end{cases} \tag{45}$$

The solutions of the above problems can be obtained by first solving the self-contained equation (45), and then plugging $\widehat{C}_{1,m}(t)$ into the equation (44) to derive $\widehat{C}_{0,m}(t)$.

Intuitively, when there is no external contagion risk (i.e., $\alpha_0 = 1$), the default intensity $h_{1,m} = h_{2,m}$, which in turn implies that there should be no difference between $\widehat{C}_{0,m}(t)$ and $\widehat{C}_{1,m}(t)$. One can simply verify that $\widehat{C}_{1,m}(t)$ also solves the equation (44). So the claim that $\widehat{C}_{0,m}(t) = \widehat{C}_{1,m}(t)$ holds immediately by applying the uniqueness of the linear ordinary differential equation (44). We will turn to this spacial case shortly and use this result at the end of this section.

5.3 Reduced Value Functions

Also from the results in Section 2 and using symmetric property, the reduced value functions $\widehat{V}_{0,m}(t)$ and $\widehat{V}_{1,m}(t)$ solve the following two non-linear ordinary differential equations, respectively:

$$\begin{cases} 0 = \frac{d}{dt}\widehat{V}_{0,m}(t) + \max_{\pi} \Big(r - \pi \Big[(N-m)\widehat{\Delta}_m(t)\widehat{h}_{0,m} + \widehat{\Delta}_{0,1,m}(t)a_0\Big]\Big)\gamma\widehat{V}_{0,m}(t) \\ \qquad +(N-m)(1+\widehat{\lambda})\widehat{h}_{0,m}\Big([1+\pi\,\widehat{\Delta}_m(t)]^{\gamma}\widehat{V}_{0,m+1}(t) - \widehat{V}_{0,m}(t)\Big) \\ \qquad +(1+\lambda_0)a_0\Big([1+\pi\,\widehat{\Delta}_{0,1,m}(t)]^{\gamma}\widehat{V}_{1,m}(t) - \widehat{V}_{0,m}(t)\Big), \quad t \in [0,T)\ , \\ \widehat{V}_{0,m}(T) = \frac{1}{\gamma}, \quad 0 \le m < k \le N, \\ \widehat{V}_{0,k}(t) = \frac{1}{\gamma}e^{r\gamma(T-t)}\ , \end{cases} \tag{46}$$

$$\begin{cases} 0 = \frac{d}{dt}\widehat{V}_{1,m}(t) + \max_{\pi} \Big(r - \pi\,(N-m)\widehat{\Delta}_{1,m}(t)\widehat{h}_{1,m}\Big)\gamma\widehat{V}_{1,m}(t) \\ \qquad +(N-m)(1+\widehat{\lambda})\widehat{h}_{1,m}\Big([1+\pi\,\widehat{\Delta}_{1,m}(t)]^{\gamma}\widehat{V}_{1,m+1}(t) - \widehat{V}_{1,m}(t)\Big), \ t \in [0,T)\ , \\ \widehat{V}_{1,m}(T) = \frac{1}{\gamma}, \quad 0 \le m < k \le N, \\ \widehat{V}_{1,k}(t) = \frac{1}{\gamma}e^{r\gamma(T-t)}\ , \end{cases} \tag{47}$$

where $\widehat{\Delta}_{0,1,m}(t)$, $\widehat{\Delta}_{0,m}(t)$ and $\widehat{\Delta}_{1,m}(t)$ are given respectively by

$$\widehat{\Delta}_{0,1,m}(t) := \frac{\widehat{C}_{1,m}(t)}{\widehat{C}_{0,m}(t)} - 1, \quad \widehat{\Delta}_{0,m}(t) := \frac{\widehat{C}_{0,m+1}(t)}{\widehat{C}_{0,m}(t)} - 1, \quad \widehat{\Delta}_{1,m}(t) := \frac{\widehat{C}_{1,m+1}(t)}{\widehat{C}_{1,m}(t)} - 1\ .$$

Likewise, to derive the solutions of the above problems, we can first solve the equation (47) and then the equation (46). Again, in the absence of external contagion risk (i.e., $\alpha_0 = 1$), $\widehat{V}_{0,m}(t)$ and $\widehat{V}_{1,m}(t)$ must be the same. To see this, we use the result $\widehat{C}_{0,m}(t) = \widehat{C}_{1,m}(t)$ to derive that $\widehat{\Delta}_{0,1,m}(t) = 0$ and $\widehat{\Delta}_{0,m}(t) = \widehat{\Delta}_{1,m}(t)$. Next, one can verify that $\widehat{V}_{0,m} = \widehat{V}_{1,m}$ is a solution of the equation (46). Finally, the uniqueness of the equation (46) guarantees the claim. In the next subsection, we will further show that the optimal investment strategies can be derived explicitly in the absence of external contagion risks.

5.4 Absence of External Contagion Risks

Since it does not matter whether or not the external shock is realized in the absence of external contagion risks (i.e., $\alpha_0 = 1$), we use the following notations to make a further simplification. Let

$$\widehat{C}_m(t) := \widehat{C}_{0,m}(t) = \widehat{C}_{1,m}(t)\ , \tag{48}$$

$$\widehat{V}_m(t) := \widehat{V}_{0,m}(t) = \widehat{V}_{1,m}(t)\ . \tag{49}$$

Then, $\widehat{C}_m$ solves the following linear ODE:

$$\begin{cases} 0 = \frac{d}{dt}\widehat{C}_m(t) + \Big((N-m)\widehat{h}_m + r\Big)\widehat{C}_m(t) \\ \qquad -(N-m)\widehat{h}_m\widehat{C}_{m+1}(t) - \kappa, \quad t \in [0,\overline{T}) \ , \\ \widehat{C}_m(\overline{T}) = 1, \quad 0 \le m < k \le N, \\ \widehat{C}_k(t) = \widehat{R} \ , \end{cases} \tag{50}$$

and $\widehat{V}_m$ solves the following nonlinear ODE:

$$\begin{cases} 0 = \frac{d}{dt}\widehat{V}_m(t) + \max_{\pi}\Big(r - \pi\,(N-m)\widehat{\Delta}_m(t)\widehat{h}_m\Big)\gamma\widehat{V}_m(t) \\ \qquad +(N-m)(1+\widehat{\lambda})\widehat{h}_m\Big([1+\pi\,\widehat{\Delta}_m(t)]^{\gamma}\widehat{V}_{m+1}(t) - \widehat{V}_m(t)\Big), \quad t \in [0,T) \ , \\ \widehat{V}_m(T) = \frac{1}{\gamma}, \quad 0 \le m < k \le N, \\ \widehat{V}_k(t) = \frac{1}{\gamma}e^{r\gamma(T-t)} \ . \end{cases} \tag{51}$$

where $\widehat{h}_m = \widehat{a}\,(\widehat{\alpha})^m$, and $\widehat{\Delta}_m(t) = \widehat{C}_{m+1}(t)/\widehat{C}_m(t) - 1$.

As a result, from the first order condition (31), we can explicitly solve for the optimal investment strategy:

$$\widehat{\pi}^*_m(t) := \frac{\Big((1+\widehat{\lambda})\widehat{V}_{m+1}(t)/\widehat{V}_m(t)\Big)^{\frac{1}{1-\gamma}} - 1}{\widehat{\Delta}_m(t)} \ . \tag{52}$$

The following proposition shows that the optimal long or short strategy is determined by the sign of default risk premium.

Theorem 2 (Long/Short Strategy)**.** *Assume that all reference entities have identical characteristics and that there is no external contagion risk. Further assume that $\kappa \ge r\widehat{R}$. Then the investor will optimally long/short the CLN if the default risk premium is negative/positive, i.e.,*

$$\begin{cases} \widehat{\pi}^*_m(t) > 0 & \textit{if } \ \widehat{\lambda} \in (-1,0) \ , \\ \widehat{\pi}^*_m(t) = 0 & \textit{if } \ \widehat{\lambda} = 0 \ , \\ \widehat{\pi}^*_m(t) < 0 & \textit{if } \ \widehat{\lambda} \in (0,\infty) \ , \end{cases}$$

where $\widehat{\pi}^*_m(t)$ is defined by (52)*.*

This long/short strategy is a reminiscence of the classic Merton (1969)'s strategy for the portfolio selection problem between risk-free bond and a risky stock, where the investor will optimally long/short the stock if the risk premium is positive/negative. Here, since a CLN is essentially a fixed income product, its default risk premium is similar to that of a default bond, whose sign is negative for a positive compensation.

The condition that $\kappa \geq r\widehat{R}$ may be a little strange at the first glance. To see its economic meaning, we can use it to derive that

$$\int_0^\infty \kappa\, e^{-rt} dt \geq \int_0^\infty r\widehat{R}\, e^{-rt} dt = \widehat{R}\ .$$

The above inequality essentially states that receiving the regular coupon payment is more preferable than taking the one-time recovery compensation conditional on an immediately termination of the CLN, which is a reasonable assumption in practice.

Now let's end this section with the proof of Proposition 2.

Proof. We split the proof into two parts: (i) $\widehat{\Delta}_m(t) < 0$, (ii) $(1+\widehat{\lambda})\widehat{V}_{m+1}(t)/\widehat{V}_m(t) < 1$ if $\widehat{\lambda} \in (-1, 0)$, $(1+\widehat{\lambda})\widehat{V}_{m+1}(t)/\widehat{V}_m(t) = 1$ if $\widehat{\lambda} = 0$, and $(1+\widehat{\lambda})\widehat{V}_{m+1}(t)/\widehat{V}_m(t) > 1$ if $\widehat{\lambda} \in (0, \infty)$, for $0 \leq m \leq k-1$.

Step (i). We claim that $\widehat{\Delta}_m(t) < 0$ or equivalently $\widehat{C}_m(t) > \widehat{C}_{m+1}(t) > 0$. Letting $\tau = \overline{T} - t$, then the linear ODE (50) can be transformed to the following initial problem

$$\frac{d}{d\tau}\widehat{C}_m(\tau) = f_m(\widehat{C}_m, \widehat{C}_{m+1}, \tau), \quad \widehat{C}_m(0) = 1\ , \tag{53}$$

where

$$f_m(\widehat{C}_m, \widehat{C}_{m+1}, \tau) := (N-m)\widehat{h}_m\big(\widehat{C}_{m+1}(\tau) - \widehat{C}_m(\tau)\big) + \kappa - r\widehat{C}_m(\tau)\ .$$

Next, we use a backward induction argument. Consider first the linear ODE (53) when $m = k-1$, where $\widehat{C}_{m+1}(\tau) = \widehat{C}_k(\tau) = \widehat{R}$. A simple calculation shows that $\widehat{C}_k(0) = \widehat{R} < 1 = \widehat{C}_{k-1}(0)$, and

$$\frac{d}{dt}\widehat{C}_k(\tau) = 0 \leq \kappa - r\widehat{R} = f(\widehat{C}_k(\tau), \widehat{C}_k(\tau), \tau)\ .$$

Hence, $\widehat{C}_k(\tau)$ is a subsolution of linear ODE (50) when $m = k-1$. Consequently, $\widehat{C}_{k-1}(\tau) > \widehat{C}_k(\tau) = \widehat{R} > 0$, which is equivalent to that $\widehat{\Delta}_{k-1}(t) < 0$.

Now, suppose that the claim holds for $m = k-1, k-2, \cdots, l+1$, i.e., $\widehat{C}_m(\tau) > C_{m+1}(\tau) > 0$, where $l \geq 0$. Consider $m = l$ and define $W(\tau) = \widehat{C}_m(\tau) - \widehat{C}_{m+1}(\tau)$. Then, we have $W(0) = 0$, and

$$\frac{d}{d\tau}W(\tau) = -\Big((N-m)\widehat{h}_m + r\Big)W(\tau) + (N-m-1)\widehat{h}_{m+1}\Big(\widehat{C}_{m+1}(\tau) - \widehat{C}_{m+2}(\tau)\Big)\ .$$

Since $\widehat{C}_{m+1}(\tau) \geq \widehat{C}_{m+2}(\tau)$ by induction, we immediately obtain $W(\tau) = \widehat{C}_m(\tau) - \widehat{C}_{m+1}(\tau) \geq 0$. As a consequence, $\widehat{\Delta}_m(\tau) < 0$ when $m = l$. This completes the proof of the Step (i).

Step (ii). We claim that $(1+\widehat{\lambda})\widehat{V}_{m+1}(t)/\widehat{V}_m(t) < 1$ if $\widehat{\lambda} \in (-1,0)$, and $(1+\widehat{\lambda})\widehat{V}_{m+1}(t)/\widehat{V}_m(t) > 1$ if $\widehat{\lambda} \in (0,\infty)$. To this end, we first plug the optimal $\widehat{\pi}^*_m$ defined (52) into the equation (51) and change the variable by $\tau = T - t$, which yields that

$$\frac{d}{d\tau}\widehat{V}_m(\tau) = g_m(\widehat{V}_m(\tau), \widehat{V}_{m+1}(\tau), \tau) \ , \quad \widehat{V}_m(0) = \frac{1}{\gamma} \ , \tag{54}$$

where

$$\begin{aligned} g_m(\widehat{V}_m(\tau), \widehat{V}_{m+1}(\tau), \tau) \ &= \ \Big(r\gamma - (N-m)\widehat{\lambda}\widehat{h}_m\Big)\widehat{V}_m(\tau) \\ &\quad + (N-m)(1-\gamma)\widehat{h}_m\bigg(\Big[(1+\widehat{\lambda})\frac{\widehat{V}_{m+1}(\tau)}{\widehat{V}_m(\tau)}\Big]^{\frac{1}{1-\gamma}} - 1\bigg)\widehat{V}_m(\tau) \ , \end{aligned}$$

for $0 \le m < k \le N$. Next, consider three subcases: (a) $\widehat{\lambda} \in (-1,0)$, (b) $\widehat{\lambda} = 0$, and (c) $\widehat{\lambda} \in (0,\infty)$.

(a) When $\widehat{\lambda} \in (-1,0)$, we claim that $\widehat{V}_m(\tau) > (1+\widehat{\lambda})\widehat{V}_{m+1}(\tau) > 0$. We again use a backward induction argument. Consider first $m = k-1$. Then, $\widehat{V}_k(\tau) = \frac{1}{\gamma}e^{r\gamma\tau}$ and

$$\begin{aligned} \frac{d}{d\tau}(1+\widehat{\lambda})\widehat{V}_k(\tau) \ &= \ r\gamma(1+\widehat{\lambda})V_k(\tau) \\ &\le \ \Big(r\gamma - (N-m)\widehat{\lambda}\widehat{h}_m\Big)(1+\widehat{\lambda})\widehat{V}_k(\tau) \\ &= \ g_m\Big((1+\widehat{\lambda})\widehat{V}_k(\tau), \widehat{V}_{m+1}(\tau), \tau\Big) \ , \end{aligned}$$

since $\widehat{\lambda} < 0$. In addition,

$$(1+\widehat{\lambda})\widehat{V}_k(0) = \frac{1+\widehat{\lambda}}{\gamma} \le \frac{1}{\gamma} = \widehat{V}_m(0) \ .$$

Consequently, $(1+\widehat{\lambda})\widehat{V}_k(\tau)$ is a subsolution of the ODE (54), which leads to $\widehat{V}_m(\tau) > (1+\widehat{\lambda})\widehat{V}_{m+1}(\tau) > 0$ when $m = k-1$. The above method can be also applied to the case where $m < k-1$. Hence we obtain that when $\widehat{\lambda} \in (-1,0)$, $\widehat{V}_m(t) > (1+\widehat{\lambda})\widehat{V}_{m+1}(t) > 0$. Combing the result in Step (i) that $\widehat{\Delta}_m(t) < 0$, we have $\widehat{\pi}^*_m(t) > 0$ when $\widehat{\lambda} \in (-1,0)$.

Next, we turn to the subcases (b) and (c). Following the same arguments as in subcase (a), we obtain that $(1+\widehat{\lambda})\widehat{V}_{m+1}(t)$ is a supersolution of the ODE (54) when $\widehat{\lambda} \in (0,\infty)$, which leads to $(1+\widehat{\lambda})\widehat{V}_{m+1}(t) \ge \widehat{V}_m(t)$. On the other hand, it is easy to see that 0 is a subsolution of the ODE (54). Hence, we obtain the require inequalities. Combing the result in Step (i) that $\widehat{\Delta}_m(t) < 0$, we have $\widehat{\pi}^*_m(t) < 0$ when $\widehat{\lambda} \in (0,\infty)$. Finally, the inequalities become equalities when $\widehat{\lambda} = 0$. This completes the proof. $\square$

6 Numerical Analysis

In this section, we turn to an extensive numerical analysis to the portfolio selection problem with a kth-to-default CLN. To focus on the effect of multiple-default protection, we begin with the symmetric case with neither internal nor external contagion risks. Then, we investigate the impact of internal and/or one-time external contagion risks. Some additional results are presented in Appendix A.

6.1 Baseline Parameter Values

Table 1: Baseline Parameter Values

Parameter Name	Symbol	Value
risk aversion factor	γ	0.5
risk free rate	r	0.03
investment horizon	T	0.9
maturity of CLN	$\overline{T}$	1
number of reference entities	N	10
number of default protection	k	4
coupon rate	κ	0.1
recovery rate	$\widehat{R}$	0.4
base default intensity	$\widehat{a}$	0.2
intensity of external shock	a_0	0.05
internal contagion risk factor	$\widehat{\alpha}$	1
external contagion risk	α_0	1
default risk premium	$\widehat{\lambda}$	-0.5
risk premium of the external shock	λ_0	-0.5

For easy exposition and comparison, we consider a symmetric case with $N = 10$ reference entities and $k = 4$ default protections. Other parameter values are specified as follows. Following Bo and Capponi (2016), we set the risk-free rate $r = 0.03$, an investor's risk aversion parameter $\gamma = 0.5$, the investment horizon $T = 0.9$ year, the maturity of the CLN $\overline{T} = 1$ year, the default risk premium $\widehat{\lambda} = -0.5$. Following Jiang, Qian, and Yuan (2017), we assume that the coupon rate $\kappa = 0.1$, the recovery rate

$\widehat{R} = 0.4$. The base default intensity is fixed at $\widehat{a} = 0.2$, which is used in Leung and Yue (2009). For the one-time external shock, as we want to intentionally model it as a rare event such as a financial crisis, the intensity of the external shock is assumed to be small $a_0 = 0.05$ also used in Leung and Yue (2009). That is, it will take about 20 years on average to have an external shock. For simplicity, the associated external risk premium factor λ_0 is assumed to have the same level as the default risk premium, i.e., $\lambda_0 = \widehat{\lambda} = -0.5$. Finally, we rule out both internal and external contagion risks at this moment so that $\widehat{\alpha} = \alpha_0 = 1$. For easy reference, all these parameter values are collected in Table 1.

We first examine the optimal investment strategies.

6.2 Optimal Investment Strategies

Recall that there are total k default protections for N reference entities. Once there are $m \leq k$ defaults, only $N - m$ entities alive in the reference pool. Hence, $\widehat{\pi}^*_m$ stands for the optimal investment in the CLN when m defaults have already occurred. That is, there are $N - m$ default protections remaining. In Figure 2, we plot the optimal investment strategies $\widehat{\pi}^*_m$ and the proportional jump sizes of the CLN $\widehat{\Delta}_m$ for three sets of parameter values: (a) the baseline parameter values (i.e., $N = 10$, $k = 4$, and $\widehat{R} = 0.4$), (b) a low scale of reference pool (i.e., $N = 6$, $k = 4$, and $\widehat{R} = 0.4$), and (c) a low recovery rate (i.e., $N = 10$, $k = 4$, and $\widehat{R} = 0.05$). Other parameter values are documented in Table 1. Several interesting observations are in order.

First, consisting with our theoretical results in Proposition 2, the investor will always take long positions (i.e., $\widehat{\pi}^*_m > 0$ for all m) when the default risk premium parameter is negative (i.e., $\widehat{\lambda} = -0.5$ meaning a positive risk compensation for a fixed income product) as illustrated in Panels A1, A2, and A3, where the dashed, dotted, dash-dotted, and solid lines plot the optimal investment strategies $\widehat{\pi}^*_m$ when m defaults have already occurred for $m = 0, 1, 2$, and 3, respectively. In Figure 6 in Appendix A, we also plot the optimal investment strategies when $\widehat{\lambda} = 0$ and $\widehat{\lambda} = 0.5$ to further verify the results in Proposition 2.

Second, the optimal investment strategy is monotonically decreasing with respect to the investment horizon when there is only one default protection remaining as presented by the solid line (i.e., $\widehat{\pi}^*_3$) in Panel A1 of Figure 2. Intuitively, as the investment horizon becomes shorter, the expected income from the coupon payment decreases, and the expected default loss is thereby increasing, which is measured by $\widehat{\Delta}_3$ in Panel A2 of

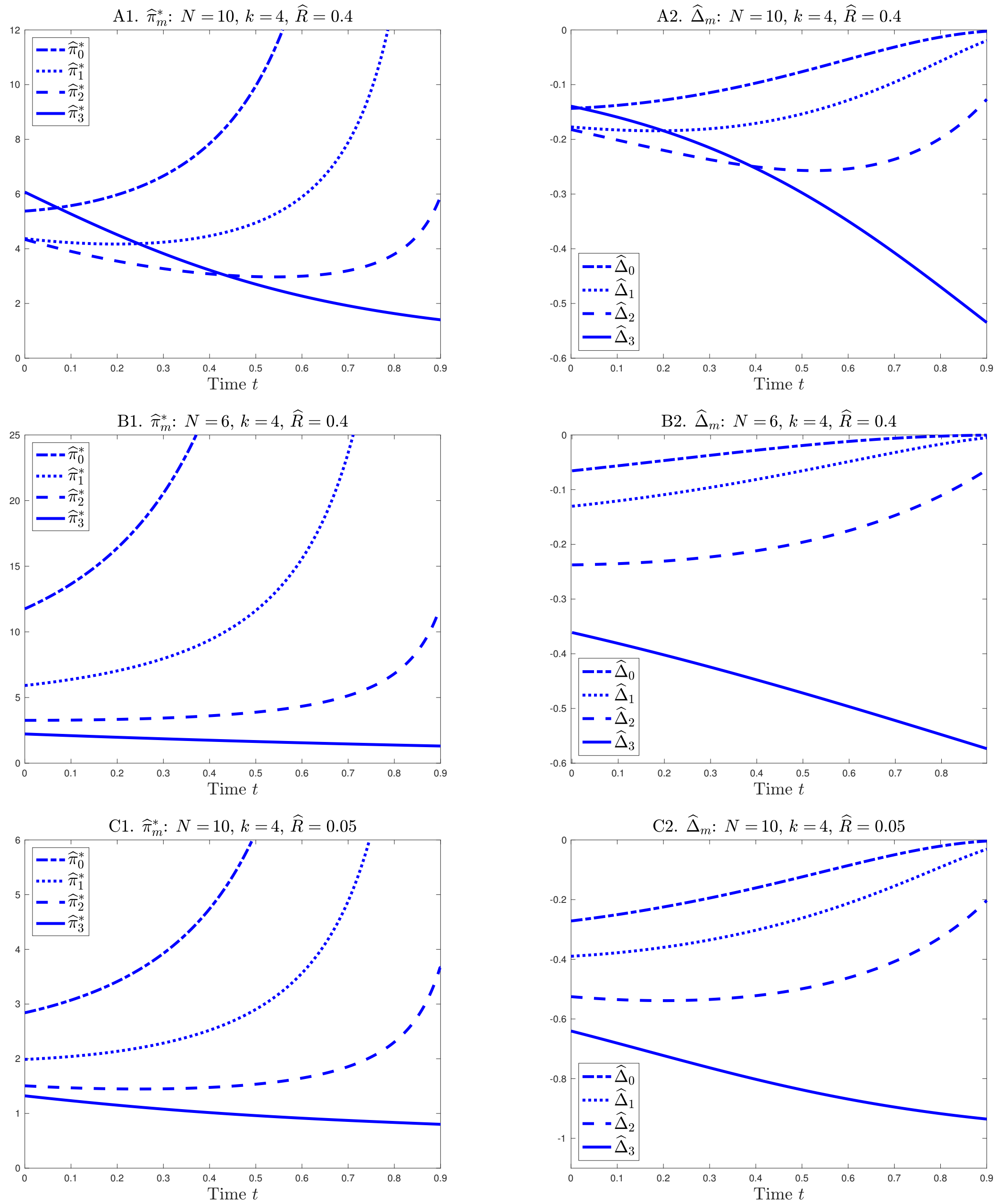

Figure 2: The optimal investment strategies $\widehat{\pi}^*_m$ and the jumps $\widehat{\Delta}_m$ with respect to the investment horizon. In Panels A1, B1, and C1, the dashed, dotted, dash-dotted, and solid lines plot the optimal investment strategies $\widehat{\pi}^*_m$ when m defaults have already occurred for $m = 0, 1, 2,$ and 3, respectively. In Panels A2, B2, and C2, the dashed, dotted, dash-dotted, and solid lines plot the proportional jump sizes of the CLN $\widehat{\Delta}_m$ when m defaults have already occurred for $m = 0, 1, 2,$ and 3, respectively. Panels A1 and A2 are for our baseline case, Panels B1 and B2 are for the case with a small scale of reference pool (i.e., $N = 6$), and Panels C1 and C2 are for the case with a small recovery rate (i.e., $\widehat{R} = 0.05$). Other parameter values are listed in Table 1.

Figure 2. As a result, the investor will invest less in the CLN to reduce her risk exposure as the investment horizon becomes short. This result for $\widehat{\pi}^*_3$ is robust for a low scale of reference pool and/or a low recovery rate as illustrated by the solid lines in Panels B1 and C1.

Third, the optimal investment strategies may be non-monotonic when there are multiple default protections; see the dashed, dotted, and dot-dashed lines for $\widehat{\pi}_m^*$ for $m = 0, 1$, and 2 in Panel A1. For a short investment horizon, the investor will take a more aggressive strategy by investing more in the CLN, which is in a sharp contrast to the decreasing pattern of investment for the last default protection case. This is because an extra default protection significantly reduces the expected default loss when the investment horizon is short as measured by $\widehat{\Delta}_m$ for $m = 0, 1, 2$ in Panel A2 of Figure 2. For a long investment horizon, however, this increasing pattern of $\widehat{\pi}_m^*$ will become weak or even reversed. The reasons are as follows. On one hand, as the investment horizon increases, the marginal value of an additional default protection tends to decrease. And this effect is more salient especially when the scale of reference pool is large. On the other hand, when the investment horizon is sufficiently long (e.g., $t < 0.1$) and the remaining number of default protections is small (e.g., $\widehat{\pi}_2$), the investor is more likely to obtain a recovery compensation of the CLN. If the recovery rate $\widehat{R}$ is sufficiently large, this compensation can generate a decreasing pattern of the expected loss as the investment horizon increases; see the dashed line (i.e., $\widehat{\Delta}_2$) in Panel A2 for illustration. Consequently, this may in turn lead to more investments in the CLN as indicated by the dashed line (i.e., $\widehat{\pi}_2$) in Panel A1.

Fourth, an additional default protection may not lead to a more investment in the CLN when the investment horizon is long. In particular, Panel A1 shows that the optimal investment strategy with one default protection $\widehat{\pi}_3^*$ (the solid line) is higher than those with multiple default protections at the early stage of the investment (i.e., $t < 0.1$). This is because, compared to the case with multiple default protections, the expected default compensation is highest for the last default protection case. Or equivalently, the expected loss from default is small as measured by $\widehat{\Delta}_m$ in Panel A2. Two key factors have critical impacts on this expected recovery compensation. The first is the likelihood of default. When the scale of the reference pool is small, *ceteris paribus*, the default probability will also be small, which in turn leads to a small expected recovery compensation. Panel B2 shows that when the scale of reference pool reduces to $N = 6$ from $N = 10$, the expected loss is largest for $\widehat{\Delta}_3$. Hence, the investor will invest least in the CLN when there is only one default protection remaining as illustrated by $\widehat{\pi}_3^*$ in Panel B1. The other factor is the recovery rate. A lower recovery rate naturally results in a lower expected value of recovery compensation. In Panel C1, we can also generate a lowest investment strategy $\widehat{\pi}_3^*$ among all investment strategies by intentionally decreasing the recovery rate to $\widehat{R} = 0.05$ from $\widehat{R} = 0.4$. Panel C2 also confirms that $\widehat{\Delta}_3$ is the largest loss.

In sum, the optimal investment strategy is essentially determined by a risk-return tradeoff. When the investment horizon is short, the extra protection plays a dominant role and gives the investor a strong incentive to invest more in the CLN. When the investment horizon is long, the expected recovery compensation becomes more important and the situation is much more complicated. The investor needs to do a careful calculation to balance the risk and return.

Next, we turn to measuring values of default protections.

6.3 Values of Default Protections

To further quantify the value of an additional default protection, we define the *proportional certainty equivalent wealth* $\varepsilon_{N,k}$ as follows:

$$v(0, w, \mathbf{0}; N, k+1) = v(0, w(1 + \varepsilon_{N,k}), \mathbf{0}; N, k) \ , \tag{55}$$

where $v(\cdot; N, k)$ is the value function defined in (22) when there is a kth-to-default CLN with N reference entities. Intuitively, $\varepsilon_{N,k}$ is the proportional compensation of initial wealth for an investor to forgive an additional default protection when there are k default protections for the N reference entities. By homogeneity and symmetry, $\varepsilon_{N,k}$ in (55) can be expressed more explicitly

$$\varepsilon_{N,k} = \Big(\frac{\widehat{V}_0(0; N, k+1)}{\widehat{V}_0(0; N, k)}\Big)^{\frac{1}{\gamma}} - 1 \ ,$$

where $\widehat{V}_0(\cdot; N, k)$ is the reduced value function defined in (49) when there is a kth-to-default CLN with N reference entities.

In Figure 3, Panel A plots $\varepsilon_{N,k}$ with respect to the number of reference entities N when the current number of default protections is fixed at $k = 1$. Other parameter values are reported in Table 1. Quantitatively, $\varepsilon_{N,k}$ increases sharply from about 1% to over 18% when the number of reference entities N increases from 2 to 10. That is, when there are 10 reference entities, the investor will need more than 18% initial wealth compensation to forgive the second default protection. In contrast, the value of an additional default protection drops dramatically from about 19% to almost zero when the number of existing default protection k increases from 1 to 9 as illustrated by Panel B in Figure 3.

So far, we have only considered the case with multiple default protections, we next investigate the impact of internal contagion risks on the optimal investment strategies.

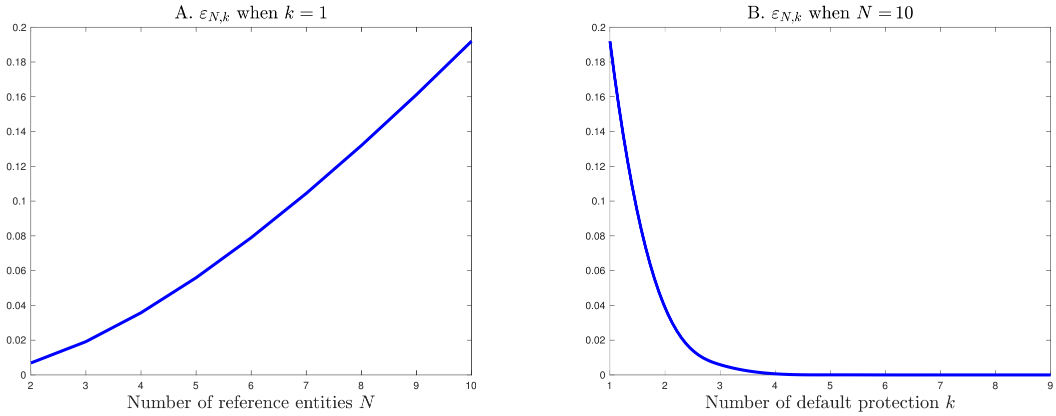

Figure 3: VALUE OF AN ADDITIONAL DEFAULT PROTECTION $\varepsilon_{N,k}$. Panel A plots $\varepsilon_{N,k}$ with respect to the number of reference entities N when the current number of default protections is fixed at $k = 1$. Panel B plots $\varepsilon_{N,k}$ with respect to the number of default protections k when the number of reference entities is fixed at $N = 10$. Other parameter values are listed in Table 1.

6.4 Internal Contagion Risk

In each panel of Figure 4, the dashed, solid, and the dotted lines are the optimal investment strategies $\widehat{\pi}_m^*$ for the cases with negative ($\widehat{\alpha} = 0.8$), zero ($\widehat{\alpha} = 1$), and positive ($\widehat{\alpha} = 1.2$) internal contagion risks, respectively. Other parameter values are documented in Table 1.

Common wisdom may suggest that when there is a positive internal contagion risk (i.e., $\widehat{\alpha} > 1$), the investor should take a more conservative investment strategy by investing less in the CLN than the case without internal contagion risks. Panel A confirms this intuition when there are four default protections remaining as the dotted line (i.e., $\widehat{\alpha} = 1.2$) is lower than the other two lines (i.e., $\widehat{\alpha} = 1$ and $\widehat{\alpha} = 0.8$). However, as the number of default protections decreases, the above intuition no longer holds. More concretely, in Panel B, the dotted line ($\widehat{\alpha} = 1.2$) is above the solid line ($\widehat{\alpha} = 1$) at the early stage of investment (i.e., $t < 0.1$). Panel C plots the case with two default protections remaining, where the dotted line ($\widehat{\alpha} = 1.2$) is above the solid line ($\widehat{\alpha} = 1$) for an even longer early stage (i.e., $t < 0.5$). And, the solid line ($\widehat{\alpha} = 1$) is also above the dashed line ($\widehat{\alpha} = 0.8$) when $t < 0.2$. More interesting, Panel D shows that a reverse result holds when there is only one default protection remaining. That is, the investor will put more wealth in the CLN when there is a positive internal contagion risk. The reason is again due to the risk-return tradeoff. For a positive contagion risk (i.e., $\widehat{\alpha} > 1$), the default intensity will increase quickly from the base level $\widehat{a}$ to the new one $\widehat{a}\,(\widehat{\alpha})^m$ after m defaults. A higher default intensity leads to a higher expected risk compensation

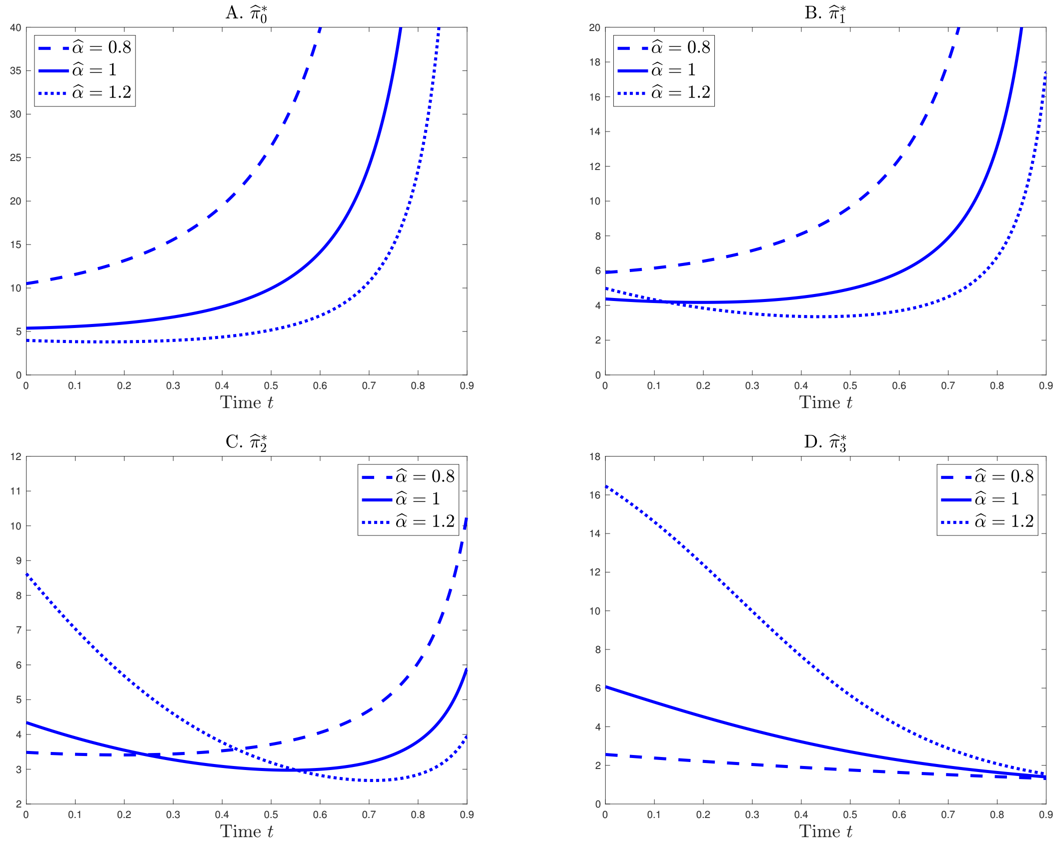

Figure 4: THE OPTIMAL INVESTMENT STRATEGIES $\widehat{\pi}_m^*$ WITH RESPECT TO INTERNAL CONTAGION RISK. In each panel of Figure 4, the dashed, solid, and the dotted lines are the optimal investment strategies $\widehat{\pi}_m^*$ for the cases with negative ($\widehat{\alpha} = 0.8$), zero ($\widehat{\alpha} = 1$), and positive ($\widehat{\alpha} = 1.2$) internal contagion risks, respectively. Other parameter values are documented in Table 1.

especially for a long investment horizon, which attracts the investor to invest more in the CLN.

Next, we examine the impact of a one-time external shock, which may have some contagion risks to the reference entities.

6.5 External Contagion Risk

To isolate the effect of external contagion risk, we turn off the internal contagion risk by setting $\widehat{\alpha} = 1$. Other parameter values are documented in Table 1. In each panel of Figure 5, the dashed, solid, and the dotted lines are the optimal investment strategies $\widehat{\pi}_m^*$ for the cases with negative ($\alpha_0 = 0.2$), zero ($\alpha_0 = 1$), and positive ($\alpha_0 = 1.8$) external contagion risks, respectively.

On one hand, comparing Figures 4 and 5, the impact of external contagion risk is qualitatively similar to the impact of internal contagion risk analyzed in the previous subsection. That is, for a positive external contagion risk (i.e., $\alpha_0 > 1$), the investor will

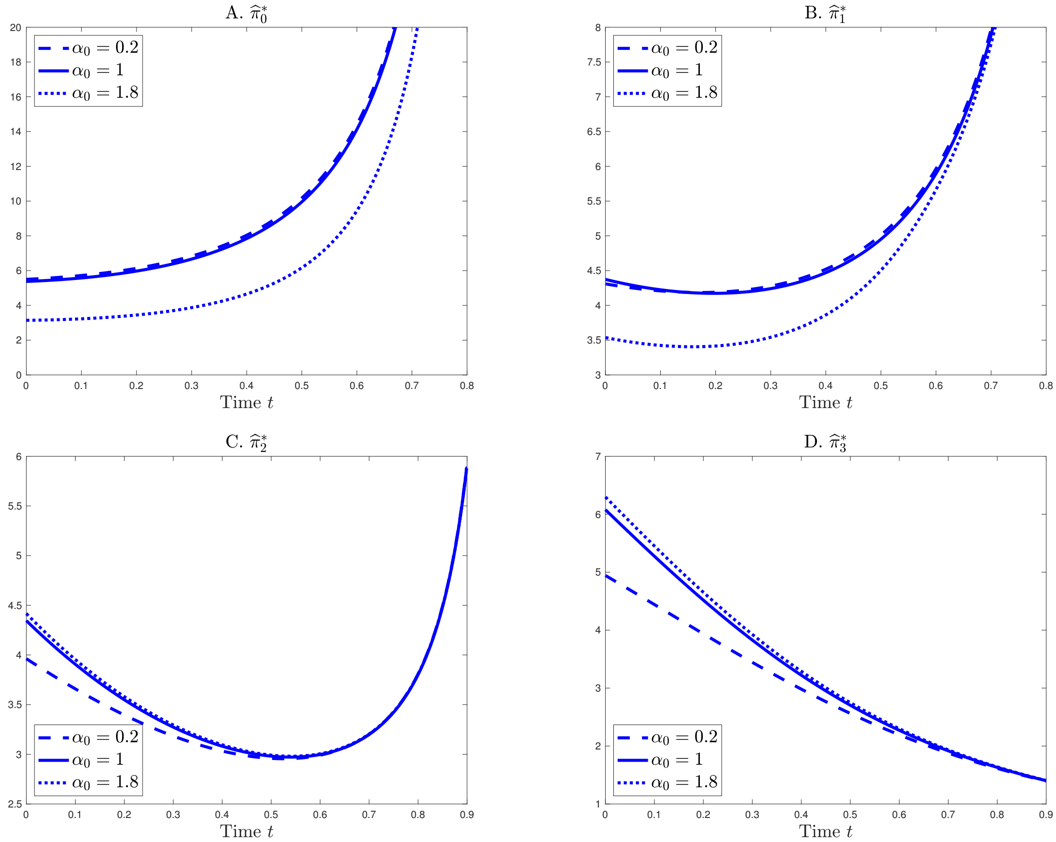

Figure 5: THE OPTIMAL INVESTMENT STRATEGIES $\widehat{\pi}_m^*$ WITH RESPECT TO EXTERNAL CONTAGION RISK. In each panel of Figure 5, the dashed, solid, and the dotted lines are the optimal investment strategies $\widehat{\pi}_m^*$ for the cases with negative ($\alpha_0 = 0.2$), zero ($\alpha_0 = 1$), and positive ($\alpha_0 = 1.8$) external contagion risks, respectively. Other parameter values are documented in Table 1.

take a more conservative investment strategy in the CLN when the external contagion risk is large and the number of default protections is large as illustrated by Panels A and B of Figure 5. However, when the number of default protections decreases, the expected compensation of default increases especially for a long investment horizon. This increasing compensation gives the investor a strong incentive to increase the wealth allocation in the CLN especially at the early stage of investment. This is why the dotted lines ($\alpha_0 = 1.8$) are above the other two lines ($\alpha_0 = 1$ and $\alpha_0 = 0.2$) when the investment horizon is sufficiently long in Panels C and D.

On the other hand, as the external shock is a one-time shock, its quantitative impact is relatively small compared to the internal contagion risk. Interestingly, from Figure 5, we find that the differences between the dotted lines ($\alpha_0 = 1.8$) and the solid lines ($\alpha_0 = 1$) become weak as the number of default protections decreases. This implies that the impact of the positive contagion risk ($\alpha_0 = 1.8$) decreases as the number of default protection decreases. In contrast, the reverse result holds for the negative contagion risk ($\alpha_0 = 0.2$). More concretely, the differences between the dashed lines ($\alpha_0 = 0.2$) and the solid lines ($\alpha_0 = 1$) become strong as the number of default protections decreases. The intuition is as follows. When the number of default protection is large, the early termination probability of the CLN is already small. Then the negative external contagion risk ($\alpha_0 < 1$), which may further reduce the default probability, has a small marginal impact for the investor. But this marginal impact will increase as the number of default protection decreases.

7 Conclusion

In this paper, we consider a portfolio selection problem of a power utility investor who optimally allocates her wealth between a risk-free bond and a kth-to-default Credit-Linked Note. In addition to multiple default protections, the CLN may have both internal and external contagion risks. The value of the CLN and its dynamics are obtained under a Markov chain model. By the dynamical programming principle, we characterize the value function as a unique classic solution to a system of Hamilton-Jacobi-Bellman equations, each of which is associated with a default or shock realization state. The optimal strategy is to make the current marginal value of wealth equal the weighted average of the risk-adjusted marginal value of wealth conditional on a default or shock realization, where the weight is determined jointly by the jump size and intensity of CLN. When all reference entities have the same characteristics and the external contagion risk is absent, we prove that the investor will take long/short positions in the CLN if the default risk compensation is positive/negative. Numerically, we find the multiple-default protection has a significant impact on optimal investment strategies. For a short investment horizon, an additional default protection leads to more investments in the CLN. However, for a long investment horizon, the CLN's early termination compensation becomes more important and may make additional default protection less attractive. This difference between short and long horizons is more salient in the presence of internal and/or external contagion risks.

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A Additional Results

In this section, we provide some further numerical results regarding the impacts of other factors to the optimal investment strategies. We begin with the discussion of default risk premium.

A.1 Impact of Default Risk Premium

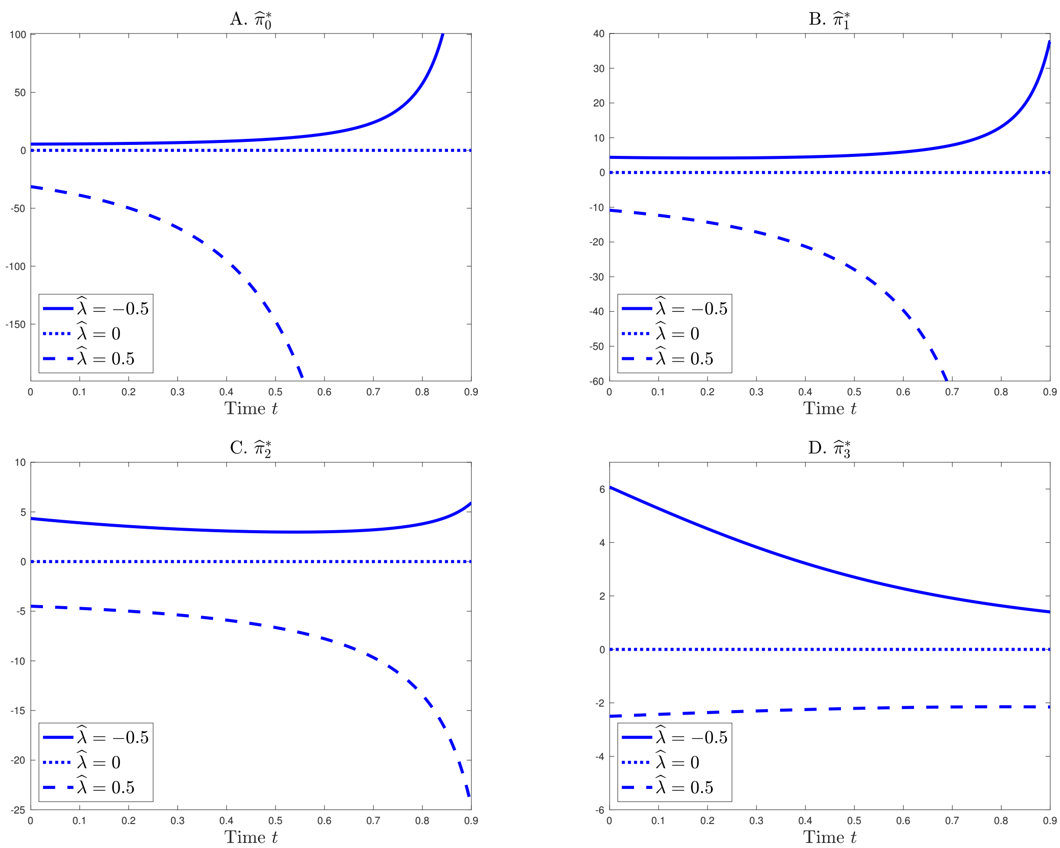

Figure 6: The optimal investment strategies $\widehat{\pi}_m^*$ with respect to default risk premium $\widehat{\lambda}$. In each panel of Figure 6, the dashed, solid, and the dotted lines are the optimal investment strategies $\widehat{\pi}_m^*$ for risk aversions $\widehat{\lambda} = -0.5$, $\widehat{\lambda} = 0$, and $\widehat{\lambda} = 0.5$, respectively. Other parameter values are documented in Table 1.

As prediction in Proposition 2, the sign of default risk premium factor $\widehat{\lambda}$ uniquely determines the direction of the investment in the CLN. In each panel of Figure 6, we confirm the theory by numerically plotting the optimal investment strategies $\widehat{\pi}_m$ for $\widehat{\lambda} = -0.5$ (the solid lines), $\widehat{\lambda} = 0$ (the dotted lines), and $\widehat{\lambda} = 0.5$ (the dashed lines), respectively. Other parameter values are summarized in Table 1.

A.2 Impact of Risk Aversion

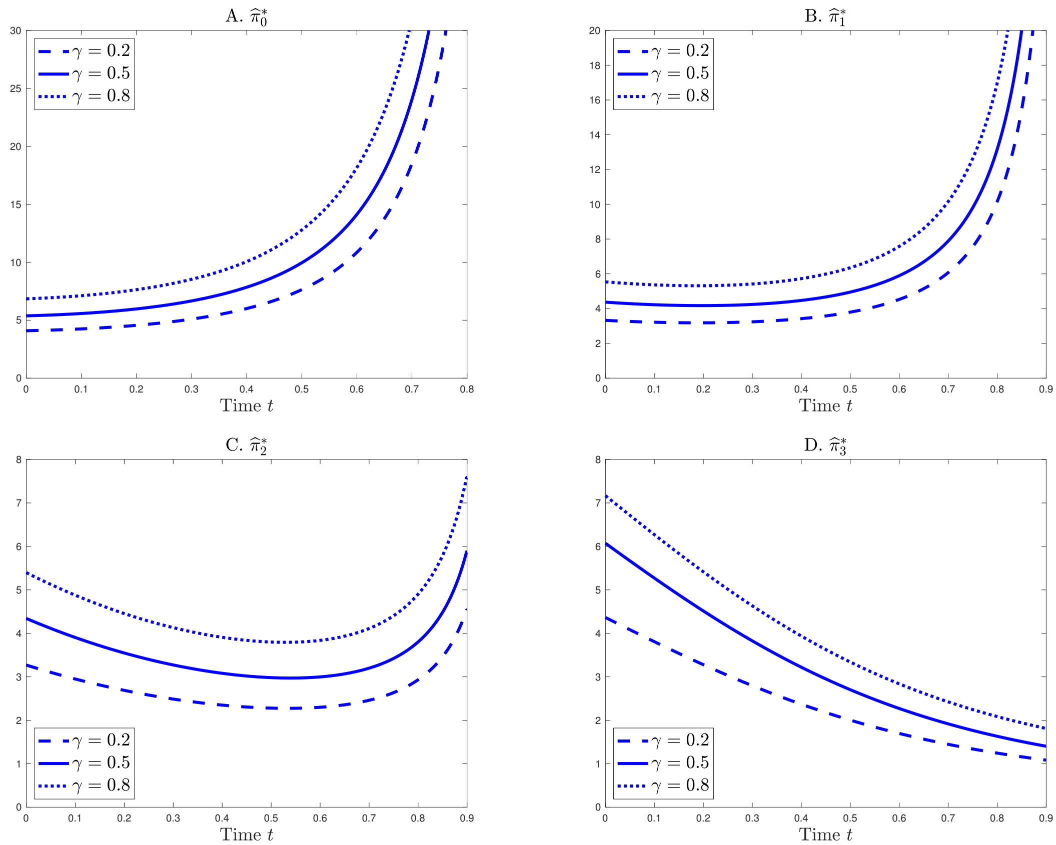

Figure 7: The optimal investment strategies $\widehat{\pi}_m^*$ with respect to risk aversions. In each panel of Figure 7, the dashed, solid, and the dotted lines are the optimal investment strategies $\widehat{\pi}_m^*$ for risk aversions $\gamma = 0.2$, $\gamma = 0.5$, and $\gamma = 0.8$, respectively. Other parameter values are documented in Table 1.

Note that $1 - \gamma$ is the investor's relative risk aversion. Intuitively, as the investor becomes more risk averse, she would invest less in the CLN. This is also verified by our numerical example in Figure 7. To be more precise, in each panel of Figure 7, the dashed, solid, and the dotted lines plot the optimal investment strategies (i.e., $\widehat{\pi}$) for $\gamma = 0.2$, $\gamma = 0.5$, and $\gamma = 0.8$, respectively. It is clear that the investment strategy is monotonically decreasing as the risk aversion $1 - \gamma$ increases.

B The value of the kth-to-default CLN

Rewrite $C^\kappa(t,\mathbf{H}(t))$, $C^i(t,\mathbf{H}(t)), i\in I\setminus M(\mathbf{z})$ and $C^L(t,\mathbf{H}(t))$ as

$$C^\kappa(t,\mathbf{H}(t)) = \mathbb{E}\left[\sum_{\substack{M_1\subset I\setminus M,\\ |M_1|\le k-1-|M|}} \int_t^{\overline{T}} \prod_{\substack{m\in M_1\\ n\in I\setminus\{M\cup M_1\}}} H_m(u)\left(1-H_n(u)\right)e^{-r(u-t)}\kappa du\middle|\mathcal{G}_t\right],$$

$$C^i(t,\mathbf{H}(t)) = \mathbb{E}\left[\sum_{\substack{M_2\subset I\setminus\{M\cup\{i\}\}\\ |M_2|=k-1-|M|}} \prod_{\substack{m\in M_2\\ n\in I\setminus\{M\cup M_2\cup\{i\}\}}} H_m(\tau_i)(1-H_n(\tau_i))\,H_i(\overline{T})\,e^{-\int_t^{\overline{T}} r(1-H_i(u))du}R_i\middle|\mathcal{G}_t\right],$$

$$C^L(t,\mathbf{H}(t)) = \mathbb{E}\left[\sum_{\substack{M_3\subset I\setminus M,\\ |M_3|\le k-1-|M|}} \prod_{\substack{m\in M_3\\ n\in I\setminus\{M\cup M_3\}}} H_m(\overline{T})(1-H_n(\overline{T}))\,e^{-r(\overline{T}-t)}\middle|\mathcal{G}_t\right],$$

We consider two cases where external Shock occurs and does not occur.

Case 1. The external shock has realized.

We use a recursive method to derive the equations satisfied by the functions $C^\kappa(t,\mathbf{z})$, $C^i(t,\mathbf{z})$ ($i\in I\setminus M(\mathbf{z})$ and $C^L(t,\mathbf{z})$ respectively.

For $C^\kappa(t,\mathbf{z})$. *Step 1.* When $|M(\mathbf{z})|=k-1$. According to the Feynman-Kac formula, the function $C^\kappa(t,\mathbf{z})$ satisfies

$$\begin{cases}\dfrac{dC^\kappa(t,\mathbf{z})}{dt} + \displaystyle\sum_{j\in I\setminus M(\mathbf{z})}(C^\kappa(t,\mathbf{z}^j)-C^\kappa(t,\mathbf{z}))h_j(\mathbf{z})+\kappa = rC^\kappa(t,\mathbf{z}),\\ C^\kappa(\overline{T},\mathbf{z})=0,\end{cases}$$

where $C^\kappa(t,\mathbf{z}^j)=0$, $j\in I\setminus M(\mathbf{z})$.

Therefore,

$$C^\kappa(t,\mathbf{z}) = \kappa\int_t^{\overline{T}} e^{-\int_t^u(r+\sum_{j\in I\setminus M(\mathbf{z})}h_j(\mathbf{z}))ds}du. \tag{B–56}$$

Step 2. When $|M(\mathbf{z})|<k-1$, the function $C^\kappa(t,\mathbf{z})$ satisfies

$$\begin{cases}\dfrac{dC^\kappa(t,\mathbf{z})}{dt} + \displaystyle\sum_{j\in I\setminus M(\mathbf{z})}(C^\kappa(t,\mathbf{z}^j)-C^\kappa(t,\mathbf{z}))h_j(\mathbf{z})+\kappa = rC^\kappa(t,\mathbf{z}),\\ C^\kappa(\overline{T},\mathbf{z})=0,\end{cases}$$

where $C^\kappa(t,\mathbf{z}^j)$ can be obtained from the condition that $|M(\mathbf{z})|+1$, $j\in I\setminus M(\mathbf{z})$.

Therefore,

$$C^\kappa(t,\mathbf{z}) = \int_t^{\overline{T}}\left(\kappa+\sum_{j\in I\setminus M(\mathbf{z})}h_j(\mathbf{z})C^\kappa(u,\mathbf{z}^j)\right)e^{-\int_t^u(r+\sum_{j\in I\setminus M(\mathbf{z})}h_j(\mathbf{z}))ds}du. \tag{B–57}$$

For $C^i(t,\mathbf{z})$. *Step 1.* When $|M(\mathbf{z})| = k-1$. If $i \in M(\mathbf{z})$, $C^i(t,\mathbf{z}) = 0$. If $i \notin M(\mathbf{z})$, the function $C^i(t,\mathbf{z})$ satisfies

$$\begin{cases} \dfrac{dC^i(t,\mathbf{z})}{dt} + \displaystyle\sum_{j \in I \setminus M(\mathbf{z})} (C^i(t,\mathbf{z}^j) - C^i(t,\mathbf{z}))h_j(\mathbf{z}) = rC^i(t,\mathbf{z}), \\ C^i(\overline{T},\mathbf{z}) = 0, \end{cases}$$

where

$$\begin{aligned} &C^i(t,\mathbf{z}^i) = R_i, \\ &C^i(t,\mathbf{z}^j) = 0, \quad j \in I \setminus (M(\mathbf{z}) \cup \{i\}). \end{aligned}$$

Therefore, we have

$$C^i(t,\mathbf{z}) = R_i \int_t^{\overline{T}} h_i(\mathbf{z}) e^{-\int_t^u (r + \sum_{j \in I \setminus M(\mathbf{z})} h_j(\mathbf{z}))ds} du. \tag{B–58}$$

Step 2. When $|M(\mathbf{z})| < k-1$. If $i \in M(\mathbf{z})$, $C^i(t,\mathbf{z}) = 0$. If $i \notin M(\mathbf{z})$, the function $C^i(t,\mathbf{z})$ satisfies

$$\begin{cases} \dfrac{dC^i(t,\mathbf{z})}{dt} + \displaystyle\sum_{j \in I \setminus M(\mathbf{z})} (C^i(t,\mathbf{z}^j) - C^i(t,\mathbf{z}))h_j(\mathbf{z}) = rC^i(t,\mathbf{z}), \\ C^i(\overline{T},\mathbf{z}) = 0, \end{cases}$$

where $C^i(t,\mathbf{z}^i) = 0$, $C^i(t,\mathbf{z}^j)$ can be obtained from the condition that $|M(\mathbf{z})| + 1$, $j \in I \setminus (M(\mathbf{z}) \cup \{i\})$.

Therefore, we have

$$C^i(t,\mathbf{z}) = \sum_{j \in I \setminus \{M(\mathbf{z}) \cup \{i\}\}} \int_t^{\overline{T}} h_j(\mathbf{z}) C^i(u,\mathbf{z}^j) e^{-\int_t^u (r + \sum_{j \in I \setminus M(\mathbf{z})} h_j(\mathbf{z}))ds} du. \tag{B–59}$$

For $C^L(t,\mathbf{z})$. *Step 1.* When $|M(\mathbf{z})| = k-1$, the function $C^L(t,\mathbf{z})$ satisfies

$$\begin{cases} \dfrac{dC^L(t,\mathbf{z})}{dt} + \displaystyle\sum_{j \in I \setminus M(\mathbf{z})} (C^L(t,\mathbf{z}^j) - C^L(t,\mathbf{z}))h_j(\mathbf{z}) = rC^L(t,\mathbf{z}), \\ C^L(\overline{T},\mathbf{z}) = 1, \end{cases}$$

where $C^L(t,\mathbf{z}^j(t)) = 0,$, $j \in I \setminus M(\mathbf{z})$.

Therefore,

$$C^L(t,\mathbf{z}) = e^{-\int_t^{\overline{T}} (r + \sum_{j \in I \setminus M(\mathbf{z})} h_j(\mathbf{z}))du}. \tag{B–60}$$

Step 2. When $|M(\mathbf{z})| < k-1$, $C^L(t,\mathbf{z})$ satisfies

$$\begin{cases} \dfrac{dC^L(t,\mathbf{z})}{dt} + \displaystyle\sum_{j\in I\setminus M(\mathbf{z})} (C^L(t,\mathbf{z}^j) - C^L(t,\mathbf{z}))h_j(\mathbf{z}) = rC^L(t,\mathbf{z}), \\ C^L(\overline{T},\mathbf{z}) = 1, \end{cases}$$

where $C^L(t,\mathbf{z}^j)$ can be obtained from the condition that $|M(\mathbf{z})|+1$, $j \in I \setminus M(\mathbf{z})$.

Therefore,

$$\begin{aligned} C^L(t,\mathbf{z}) =& e^{-\int_t^{\overline{T}}(r+\sum_{j\in I\setminus M(\mathbf{z})} h_j(\mathbf{z}))du} \\ &+ \sum_{j\in I\setminus M(\mathbf{z})} \int_t^{\overline{T}} h_j(\mathbf{z})C^L(u,\mathbf{z}^j)e^{-\int_t^u(r+\sum_{j\in I\setminus M(\mathbf{z})} h_j(\mathbf{z}))ds}du. \end{aligned} \tag{B–61}$$

Case 2. The external shock does not realize.

For $C^\kappa(t,\mathbf{z})$. *Step 1.* When $|M(\mathbf{z})| = k-1$, $C^\kappa(t,\mathbf{z})$ satisfies

$$\begin{cases} \dfrac{dC^\kappa(t,\mathbf{z})}{dt} + \displaystyle\sum_{j\in I\cup\{0\}} (1-z_j)(C^\kappa(t,\mathbf{z}^j) - C^\kappa(t,\mathbf{z}))h_j(\mathbf{z}) + \kappa = rC^\kappa(t,\mathbf{z}), \\ C^\kappa(\overline{T},\mathbf{z}) = 0, \end{cases}$$

where $C^\kappa(t,\mathbf{H}^j(t)) = 0$, $j \in I \setminus M(\mathbf{z})$, $C^\kappa(t,\mathbf{z}^0)$ has been obtained.

Therefore,

$$C^\kappa(t,\mathbf{z}) = \int_t^{\overline{T}} \left(\kappa + h_0(\mathbf{z})C^\kappa(u,\mathbf{z}^0)\right) e^{-\int_t^u(r+\sum_{j\in I\cup\{0\}\setminus M(\mathbf{z})} h_j(\mathbf{z}))ds}du. \tag{B–62}$$

Step 2. When $|M(\mathbf{z})| < k-1$, the function $C^\kappa(t,\mathbf{z})$ satisfies

$$\begin{cases} \dfrac{dC^\kappa(t,\mathbf{z})}{dt} + \displaystyle\sum_{j\in I\cup\{0\}} (1-z_j)(C^\kappa(t,\mathbf{z}^j) - C^\kappa(t,\mathbf{z}))h_j(\mathbf{z}) + \kappa = rC^\kappa(t,\mathbf{z}), \\ C^\kappa(\overline{T},\mathbf{z}) = 0, \end{cases}$$

where $C^\kappa(t,\mathbf{z}^j)$ $j \in I \setminus M(\mathbf{z})$ and $C^\kappa(t,\mathbf{z}^0)$ have been obtained.

Therefore, we can obtain

$$C^\kappa(t,\mathbf{z}) = \int_t^{\overline{T}} \left(\kappa + \sum_{j\in I\cup\{0\}\setminus M(\mathbf{z})} C^\kappa(u,\mathbf{z}^j)h_j(\mathbf{z})\right) e^{-\int_t^u(r+\sum_{j\in I\cup\{0\}\setminus M(\mathbf{z})} h_j(\mathbf{z}))ds}du. \tag{B–63}$$

For $C^i(t,\mathbf{z})$. *Step 1.* When $|M(\mathbf{z})| = k-1$. If $i \in I \setminus M(\mathbf{z})$, $C^i(t,\mathbf{z}) = 0$. If $i \notin I \setminus M(\mathbf{z})$, $C^i(t,\mathbf{z})$ satisfies

$$\begin{cases} \dfrac{dC^i(t,\mathbf{z})}{dt} + \displaystyle\sum_{j\in I\cup\{0\}} (1-z_j)(C^i(t,\mathbf{z}^j) - C^i(t,\mathbf{z}))h_j(\mathbf{z}) = rC^i(t,\mathbf{z}), \\ C^i(\overline{T},\mathbf{z}) = 0, \end{cases}$$

where $C^i(t,\mathbf{z}^i)=R_i$, $C^i(t,\mathbf{z}^j)=0$, $j\in I\setminus(M(\mathbf{z})\cup\{i\})$, $C^i(t,\mathbf{z}^0)$ has been obtained.

Therefore,

$$C^i(t,\mathbf{z})=\int_t^{\overline{T}}\left(R_ih_i(\mathbf{z})+h_0(\mathbf{z})C^i(u,\mathbf{z}^0)\right)e^{-\int_t^u(r+\sum_{j\in I\cup\{0\}\setminus M(\mathbf{z})}h_j(\mathbf{z}))ds}du. \tag{B–64}$$

Step 2. When $|M(\mathbf{z})|<k-1$. If $i\in I\setminus M(\mathbf{z})$, $C^i(t,\mathbf{z})=0$. If $i\notin I\setminus M(\mathbf{z})$, the function $C^i(t,\mathbf{z})$ satisfies

$$\begin{cases}\dfrac{dC^i(t,\mathbf{z})}{dt}+\displaystyle\sum_{j\in I\cup\{0\}}(1-z_j)(C^i(t,\mathbf{z}^j)-C^i(t,\mathbf{z}))h_j(\mathbf{z})=rC^i(t,\mathbf{z}),\\ C^i(\overline{T},\mathbf{z})=0,\end{cases}$$

where $C^i(t,\mathbf{z}^i)=0$, $C^i(t,\mathbf{z}^j)$ $(j\in I\setminus M(\mathbf{z}))$ and $C^i(t,\mathbf{z}^0)$ have been obtained.

Therefore,

$$C^i(t,\mathbf{z})=\sum_{j\in I\cup\{0\}\setminus\{M(\mathbf{z})\cup\{i\}\}}\int_t^{\overline{T}}h_j(\mathbf{z})C^i(u,\mathbf{z}^j)e^{-\int_t^u(r+\sum_{j\in I\cup\{0\}\setminus M(\mathbf{z})}h_j(\mathbf{z}))ds}du. \tag{B–65}$$

For $C^L(t,\mathbf{z})$. *Step 1.* When $|M(\mathbf{z})|=k-1$, the function $C^L(t,\mathbf{z})$ satisfies

$$\begin{cases}\dfrac{dC^L(t,\mathbf{z})}{dt}+\displaystyle\sum_{j\in I\cup\{0\}}(1-z_j)(C^L(t,\mathbf{z}^j)-C^L(t,\mathbf{z}))h_j(\mathbf{z})=rC^L(t,\mathbf{z}),\\ C^L(\overline{T},\mathbf{z})=1,\end{cases}$$

where $C^L(t,\mathbf{H}^j(t))=0$, $j\in I\setminus M(\mathbf{z})$. $C^L(t,\mathbf{z}^0)$ has been obtained.

Therefore,

$$\begin{aligned}C^L(t,\mathbf{z})=&e^{-\int_t^{\overline{T}}(r+\sum_{j\in I\cup\{N+1\}\setminus M(\mathbf{z})}h_j(\mathbf{z}))du}\\&+\int_t^{\overline{T}}h_0(\mathbf{z})C^L(u,\mathbf{z}^0)e^{-\int_t^u(r+\sum_{j\in I\cup\{0\}\setminus M(\mathbf{z})}h_j(\mathbf{z}))ds}du.\end{aligned} \tag{B–66}$$

Step 2. When $|M(\mathbf{z})|<k-1$, $C^L(t,\mathbf{z})$ satisfies

$$\begin{cases}\dfrac{dC^L(t,\mathbf{z})}{dt}+\displaystyle\sum_{j\in I\cup\{0\}}(1-z_j)(C^L(t,\mathbf{z}^j)-C^L(t,\mathbf{z}))h_j(\mathbf{z})=rC^L(t,\mathbf{z}),\\ C^L(\overline{T},\mathbf{z})=1,\end{cases}$$

where $C^L(t,\mathbf{z}^j)$ $(j\in I\setminus M(\mathbf{z}))$ and $C^L(t,\mathbf{z}^0)$ have been obtained.

Therefore,

$$\begin{aligned}C^L(t,\mathbf{z})=&e^{-\int_t^{\overline{T}}(r+\sum_{j\in I\cup\{0\}\setminus M(\mathbf{z})}h_j(\mathbf{z}))du}\\&+\sum_{j\in I\cup\{0\}\setminus M(\mathbf{z})}\int_t^{\overline{T}}h_j(\mathbf{z})C^L(t,\mathbf{z}^j)e^{-\int_t^u(r+\sum_{j\in I\cup\{0\}\setminus M(\mathbf{z})}h_j(\mathbf{z}))ds}du.\end{aligned} \tag{B–67}$$

By (B–56)-(B–67), we obtain the explicit expression of the value of the kth-to-default CLN,

$$C(t,\mathbf{z})=C^{\kappa}(t,\mathbf{z})+\sum_{i\in I\setminus M(\mathbf{z})}C^{i}(t,\mathbf{z})+C^{L}(t,\mathbf{z}).$$

C Proof of Proposition 2

The calculation process is divided into two parts: the first step is to calculate the dynamic of $C(t,\mathbf{H}(t))$, and the second step is to calculate the dynamic of $\widetilde{C}(t,\mathbf{H}(t))$.

Step 1. According to the Ito formula, $C^{i}(t,\mathbf{H}(t))$ $i\in I\setminus M(\mathbf{H}(t))$ satisfy

$$\begin{aligned}
&dC^{i}(t,\mathbf{H}(t))\\
=&\frac{\partial C^{i}(t,\mathbf{H}(t-))}{\partial t}dt+\sum_{j\in I\cup\{0\}\setminus M(\mathbf{H}(t))}\left(C^{i}(t,\mathbf{H}^{j}(t-))-C^{i}(t,\mathbf{H}(t-))\right)dH_{j}(t)\\
=&\frac{\partial C^{i}(t,\mathbf{H}(t-))}{\partial t}dt+\sum_{j\in I\cup\{0\}\setminus M(\mathbf{H}(t))}\left(C^{i}(t,\mathbf{H}^{j}(t-))-C^{i}(t,\mathbf{H}(t-))\right)(1-H_{j}(t-))h_{j}(\mathbf{H}(t-))dt\\
&+\sum_{j\in I\cup\{0\}\setminus M(\mathbf{H}(t))}\left(C^{i}(t,\mathbf{H}^{j}(t-))-C^{i}(t,\mathbf{H}(t-))\right)d\xi_{j}(t)\\
=&r(1-H_{i}(t-))C^{i}(t,\mathbf{H}(t-))dt+\sum_{j\in I\cup\{0\}\setminus M(\mathbf{H}(t))}\left(C^{i}(t,\mathbf{H}^{j}(t-))-C^{i}(t,\mathbf{H}(t-))\right)d\xi_{j}(t).
\end{aligned}\tag{C–68}$$

Similarly, we have

$$\begin{aligned}
dC^{\kappa}(t,\mathbf{H}(t))=&rC^{\kappa}(t,\mathbf{H}(t))dt-\kappa dt\\
&+\sum_{j\in I\cup\{0\}\setminus M(\mathbf{H}(t))}\left(C^{i}(t,\mathbf{H}^{j}(t-))-C^{i}(t,\mathbf{H}(t-))\right)d\xi_{j}(t),
\end{aligned}\tag{C–69}$$

and

$$\begin{aligned}
&dC^{L}(t,\mathbf{H}(t))\\
=&rC^{L}(t,\mathbf{H}(t))dt+\sum_{j\in I\cup\{0\}\setminus M(\mathbf{H}(t))}\left(C^{L}(t,\mathbf{H}^{j}(t-))-C^{L}(t,\mathbf{H}(t-))\right)d\xi_{j}(t).
\end{aligned}\tag{C–70}$$

According to (C–68)-(C–70), it holds that

$$\begin{aligned}
&dC(t,\mathbf{H}(t))\\
=&\left(\sum_{j\in I\setminus M(\mathbf{H}(t))}r(1-H_{j}(t-))C^{j}(t,\mathbf{H}(t-))+rC^{\kappa}(t,\mathbf{H}(t-))+rC^{L}(t,\mathbf{H}(t-))-\kappa\right)dt
\end{aligned}$$

$$+ \sum_{j\in I\cup\{0\}\setminus M(\mathbf{H}(t))} \big(C(t,\mathbf{H}^j(t-)) - C(t,\mathbf{H}(t-))\big)\, d\xi_j(t). \tag{C–71}$$

Step 2.

$$\begin{aligned}\widetilde{C}(t,\mathbf{H}(t)) =&\mathbf{1}_{\{|M(\mathbf{H}(t))|\le k-1\}}C(t,\mathbf{H}(t))\\ =&\Bigg(\sum_{\substack{M\subset I\\ |M|\le k-1}}\prod_{\substack{m\in M\\ n\in I\setminus M}} H_m(t)(1-H_n(t))\Bigg)C(t,\mathbf{H}(t))\end{aligned}$$

When $|M(\mathbf{H}(t))|\le k-1$,

$$\widetilde{C}(t,\mathbf{H}(t)) = \Bigg(\prod_{\substack{m\in M(\mathbf{H}(t))\\ n\in I\setminus M(\mathbf{H}(t))}} H_m(t)(1-H_n(t))\Bigg)C(t,\mathbf{H}(t)).$$

We calculate $\widetilde{C}(t,\mathbf{H}(t))$ in two cases.

Case 1. The external shock has occurred.

$$\begin{aligned}
&d\widetilde{C}(t,\mathbf{H}(t))\\
=&d\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t))\\ n\in I\setminus M(\mathbf{H}(t))}} H_m(t)(1-H_n(t))\Bigg)C(t,\mathbf{H}(t))\\
=&C(t,\mathbf{H}(t))d\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t))\\ n\in I\setminus M(\mathbf{H}(t))}} H_m(t)(1-H_n(t))\Bigg)\\
&+\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t))\\ n\in I\setminus M(\mathbf{H}(t))}} H_m(t)(1-H_n(t))\Bigg)dC(t,\mathbf{H}(t))\\
&+\Delta\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t))\\ n\in I\setminus M(\mathbf{H}(t))}} H_m(t)(1-H_n(t))\Bigg)\Delta C(t,\mathbf{H}(t))\\
=&\sum_{j\in M(\mathbf{H}(t-))}\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\setminus\{j\}\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)(1-H_j(t-))C(t,\mathbf{H}(t-))dH_j(t)\\
&-\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)C(t,\mathbf{H}(t-))\sum_{j\in I\setminus M(\mathbf{H}(t))}dH_j(t)\\
&+\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\\
&\qquad\times\Bigg(rC(t,\mathbf{H}(t))-\kappa-\sum_{j\in I\setminus M(\mathbf{H}(t-))}C(t,\mathbf{H}^j(t-))h_j(\mathbf{H}(t-))\Bigg)dt\\
&+\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\sum_{j\in I\setminus M(\mathbf{H}(t-))}C(t,\mathbf{H}^j(t-))dH_j(t)
\end{aligned}$$

$$
\begin{aligned}
&-\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\sum_{j\in I\setminus M(\mathbf{H}(t-))} C(t,\mathbf{H}(t-))d\xi_j(t)\\
&+\sum_{j\in M(\mathbf{H}(t-))}\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\setminus\{j\}\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)(1-H_j(t-))\\
&\qquad\times\big(C(t,\mathbf{H}^j(t-))-C(t,\mathbf{H}(t-))\big)\,dH_j(t)\\
&-\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t))\\ n\in I\setminus M(\mathbf{H}(t))}} H_m(t-)(1-H_n(t-))\Bigg)\sum_{j\in I\setminus M(\mathbf{H}(t))}\big(C(t,\mathbf{H}^j(t-))-C(t,\mathbf{H}(t-))\big)\,dH_j(t)\\
=&\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\\
&\qquad\times\Bigg(rC(t,\mathbf{H}(t-))-\kappa-\sum_{j\in I\setminus M(\mathbf{H}(t-))} C(t,\mathbf{H}^j(t-))h_j(\mathbf{H}(t-))\Bigg)dt\\
&-\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\sum_{j\in I\setminus M(\mathbf{H}(t-))} C(t,\mathbf{H}(t-))d\xi_j(t)\\
&+\sum_{j\in M(\mathbf{H}(t-))}\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\setminus\{j\}\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)(1-H_j(t-))C(t,\mathbf{H}^j(t-))dH_j(t).
\end{aligned}
$$

- If $|M(\mathbf{H}(t))|=k-1$,

$$
\begin{aligned}
&d\widetilde{C}(t,\mathbf{H}(t))\\
&=\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\\
&\quad\times\Bigg(rC(t,\mathbf{H}(t-))-\kappa-\sum_{j\in I\setminus M(\mathbf{H}(t-))} C(t,\mathbf{H}^j(t-))h_j(\mathbf{H}(t-))\Bigg)dt\\
&\quad-\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\sum_{j\in I\setminus M(\mathbf{H}(t-))} C(t,\mathbf{H}(t-))d\xi_j(t). \qquad \text{(C–72)}
\end{aligned}
$$

- If $|M(\mathbf{H}(t-))|<k-1$. Assume $j\in I\setminus M(\mathbf{H}(t))$ default,

$$
\prod_{\substack{m\in M(\mathbf{H}(t))\cup\{j\}\\ n\in I\setminus(M(\mathbf{H}(t))\cup\{j\})}} H_m(t)\,(1-H_n(t))\,C(t,\mathbf{H}(t))
$$

has a jump

$$
\Delta=\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t))\\ n\in I\setminus M(\mathbf{H}(t))}} H_m(t-)(1-H_n(t-))\Bigg)C(t,\mathbf{H}^j(t-))\Delta H_j(t).
$$

Therefore, we have

$$
\begin{aligned}
&d\widetilde{C}(t,\mathbf{H}(t))\\
=&\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\Big(rC(t,\mathbf{H}(t-))-\kappa\Big)dt\\
&+\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\\
&\qquad\times\sum_{j\in I\setminus M(\mathbf{H}(t-))}\Big(C(t,\mathbf{H}^j(t-))-C(t,\mathbf{H}(t-))\Big)\,d\xi_j(t).
\end{aligned}
\tag{C–73}
$$

Case 2. The external shock does not occur.

$$
\begin{aligned}
&d\widetilde{C}(t,\mathbf{H}(t))\\
=&d\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t))\\ n\in I\setminus M(\mathbf{H}(t))}} H_m(t)(1-H_n(t))\Bigg)C(t,\mathbf{H}(t))\\
=&C(t,\mathbf{H}(t))d\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t))\\ n\in I\setminus M(\mathbf{H}(t))}} H_m(t)(1-H_n(t))\Bigg)\\
&+\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t))\\ n\in I\setminus M(\mathbf{H}(t))}} H_m(t)(1-H_n(t))\Bigg)dC(t,\mathbf{H}(t))\\
&+\Delta\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t))\\ n\in I\setminus M(\mathbf{H}(t))}} H_m(t)(1-H_n(t))\Bigg)\Delta C(t,\mathbf{H}(t))\\
=&\sum_{j\in M(\mathbf{H}(t-))}\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\setminus\{j\}\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)(1-H_j(t-))C(t,\mathbf{H}(t-))dH_j(t)\\
&-\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)C(t,\mathbf{H}(t-))\sum_{j\in I\setminus M(\mathbf{H}(t-))}dH_j(t)\\
&+\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t)(1-H_n(t-))\Bigg)\\
&\qquad\times\Bigg(rC(t,\mathbf{H}(t-))-\kappa-\sum_{j\in I\setminus M(\mathbf{H}(t-))}C(t,\mathbf{H}^j(t-))h_j(\mathbf{H}(t-))\Bigg)dt\\
&+\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\sum_{j\in I\setminus M(\mathbf{H}(t-))}C(t,\mathbf{H}^j(t-))dH_j(t)\\
&-\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\sum_{j\in I\setminus M(\mathbf{H}(t-))}C(t,\mathbf{H}(t-))d\xi_j(t)
\end{aligned}
$$

$$+\left(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\right)\Big(C(t,\mathbf{H}^0(t-))-C(t,\mathbf{H}(t-))\Big)d\xi_0(t)$$

$$+\sum_{j\in M(\mathbf{H}(t-))}\left(\prod_{\substack{m\in M(\mathbf{H}(t-))\setminus\{j\}\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\right)(1-H_j(t-))$$

$$\times\big(C(t,\mathbf{H}^j(t-))-C(t,\mathbf{H}(t-))\big)\,dH_j(t)$$

$$-\left(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\right)\sum_{j\in I\setminus M(\mathbf{H}(t-))}\big(C(t,\mathbf{H}^j(t-))-C(t,\mathbf{H}(t-))\big)\,dH_j(t)$$

$$=\left(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\right)$$

$$\times\left(rC(t,\mathbf{H}(t-))-\kappa-\sum_{j\in I\setminus M(\mathbf{H}(t-))} C(t,\mathbf{H}^j(t-))h_j(\mathbf{H}(t-))\right)dt$$

$$-\left(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\right)\sum_{j\in I\setminus M(\mathbf{H}(t-))} C(t,\mathbf{H}(t-))d\xi_j(t)$$

$$+\left(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\right)\Big(C(t,\mathbf{H}^0(t-))-C(t,\mathbf{H}(t-))\Big)d\xi_0(t)$$

$$+\sum_{j\in M(\mathbf{H}(t-))}\left(\prod_{\substack{m\in M(\mathbf{H}(t-))\setminus\{j\}\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\right)(1-H_j(t-))C(t,\mathbf{H}^j(t-))dH_j(t).$$

- If $|M(\mathbf{H}(t-))|<k-1$,

$$d\widetilde{C}(t,\mathbf{H}(t))$$

$$=\left(\prod_{\substack{m\in M(\mathbf{H}(t))\\ n\in I\setminus M(\mathbf{H}(t))}} H_m(t)(1-H_n(t))\right)\Big(rC(t,\mathbf{H}(t))-\kappa\Big)dt$$

$$+\left(\prod_{\substack{m\in M(\mathbf{H}(t))\\ n\in I\setminus M(\mathbf{H}(t))}} H_m(t-)(1-H_n(t-))\right)\sum_{j\in I\setminus M(\mathbf{H}(t))}\Big(C(t,\mathbf{H}^j(t-))-C(t,\mathbf{H}(t-))\Big)d\xi_j(t)$$

$$+\left(\prod_{\substack{m\in M(\mathbf{H}(t))\\ n\in I\setminus M(\mathbf{H}(t))}} H_m(t-)(1-H_n(t-))\right)\Big(C(t,\mathbf{H}^0(t-))-C(t,\mathbf{H}(t-))\Big)d\xi_0(t)\ . \quad \text{(C–74)}$$

- If $|M(\mathbf{H}(t-))|=k-1$,

$$d\widetilde{C}(t,\mathbf{H}(t))$$

$$=\left(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\right)$$

$$\times\left(rC(t,\mathbf{H}(t-))-\kappa-\sum_{j\in I\setminus M(\mathbf{H}(t-))}C(t,\mathbf{H}^j(t-))h_j(\mathbf{H}(t-))\right)dt$$
$$-\left(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}}H_m(t-)(1-H_n(t-))\right)\sum_{j\in I\setminus M(\mathbf{H}(t-))}C(t,\mathbf{H}(t-))d\xi_j(t)$$
$$+\left(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}}H_m(t-)(1-H_n(t-))\right)\big(C(t,\mathbf{H}^0(t-))-C(t,\mathbf{H}(t-))\big)\,d\xi_0(t). \quad \text{(C–75)}$$

By further calculation, we can obtain the $\mathbb{Q}$-dynamics of the kth-to-default CLN value.

D Proof of Proposition 4

The calculation process is also divided into two cases: one where an external shock occurs and one where no external shock occurs.

Case 1. The external shock has realized.

$$\begin{aligned}
&dW_t(\boldsymbol{\phi})\\
=&\phi(t-)\Big(d\widetilde{C}(t,\mathbf{H}(t))+dD_t\Big)+\phi^B(t)dB_t\\
=&\phi(t-)\Bigg(\mathbf{1}_{\{|M(\mathbf{H}(t-))|<k-1\}}\Bigg(\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}}H_m(t-)(1-H_n(t-))\Bigg)\left(rC(t,\mathbf{H}(t-))-\kappa\right)dt\\
&+\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}}H_m(t-)(1-H_n(t-))\Bigg)\sum_{j\in I\setminus M(\mathbf{H}(t-))}\big(C(t,\mathbf{H}^j(t-))-C(t,\mathbf{H}(t-))\big)\,d\xi_j^{\mathbb{P}}(t)\\
&+\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}}H_m(t-)(1-H_n(t-))\Bigg)\\
&\quad\times\sum_{j\in I\setminus M(\mathbf{H}(t-))}\big(C(t,\mathbf{H}^j(t-))-C(t,\mathbf{H}(t-))\big)\big(h_j^{\mathbb{P}}(\mathbf{H}(t-))-h_j(\mathbf{H}(t-))\big)\,dt\Bigg)\\
&+\mathbf{1}_{\{|M(\mathbf{H}(t-))|=k-1\}}\Bigg(\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}}H_m(t-)(1-H_n(t-))\Bigg)\\
&\qquad\times\Bigg(rC(t,\mathbf{H}(t-))-\kappa-\sum_{j\in I\setminus M(\mathbf{H}(t-))}C(t,\mathbf{H}^j(t-))h_j(\mathbf{H}(t-))\Bigg)dt\\
&-\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}}H_m(t-)(1-H_n(t-))\Bigg)\sum_{j\in I\setminus M(\mathbf{H}(t-))}C(t,\mathbf{H}(t-))d\xi_j^{\mathbb{P}}(t)
\end{aligned}$$

$$
\begin{aligned}
&- \left(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-)) \right) \sum_{j\in I\setminus M(\mathbf{H}(t-))} C(t,\mathbf{H}(t-)) \left(h_j^{\mathbb{P}}(\mathbf{H}(t-)) - h_j(\mathbf{H}(t-))\right) dt \Bigg) \Bigg) \\
&+ \phi(t-) \Bigg(\mathbf{1}_{\{|M(\mathbf{H}(t-))|=k-1\}} \Bigg(\left(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-)) \right) \sum_{j\in I\setminus M(\mathbf{H}(t-))} R_j dH_j(t) \Bigg) \\
&+ \mathbf{1}_{|M(\mathbf{H}(t-))|\le k-1} \left(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-)) \right) \kappa dt \Bigg) \\
&+ r\phi^B(t) B_t dt \\
=&\phi(t-) \Bigg(\mathbf{1}_{\{|M(\mathbf{H}(t-))|<k-1\}} \Bigg(\left(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-)) \right) \\
&\qquad\qquad \times \sum_{j\in I\setminus M(\mathbf{H}(t-))} \left(C(t,\mathbf{H}^j(t-)) - C(t,\mathbf{H}(t-))\right) d\xi_j^{\mathbb{P}}(t) \\
&+ \left(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-)) \right) \\
&\quad \times \sum_{j\in I\setminus M(\mathbf{H}(t-))} \left(C(t,\mathbf{H}^j(t-)) - C(t,\mathbf{H}(t-))\right) \left(h_j^{\mathbb{P}}(\mathbf{H}(t-)) - h_j(\mathbf{H}(t-))\right) dt \Bigg) \\
&+ \mathbf{1}_{\{|M(\mathbf{H}(t-))|=k-1\}} \Bigg(\left(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-)) \right) \\
&\qquad\qquad \times \Bigg(- \sum_{j\in I\setminus M(\mathbf{H}(t-))} C(t,\mathbf{H}^j(t-)) h_j(\mathbf{H}(t-)) \Bigg) dt \\
&- \left(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-)) \right) \sum_{j\in I\setminus M(\mathbf{H}(t-))} \left(R_j - C(t,\mathbf{H}(t-))\right) d\xi_j^{\mathbb{P}}(t) \\
&+ \left(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-)) \right) \sum_{j\in I\setminus M(\mathbf{H}(t-))} R_j h_j^{\mathbb{P}}(\mathbf{H}(t-)) dt \\
&- \left(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-)) \right) \sum_{j\in I\setminus M(\mathbf{H}(t-))} C(t,\mathbf{H}(t-)) \left(h_j^{\mathbb{P}}(\mathbf{H}(t-)) - h_j(\mathbf{H}(t-))\right) dt \Bigg) \Bigg) \\
&+ r\phi(t) CLN_t dt + r\phi^B(t) B_t dt \\
=&rW_t dt \\
&+ \phi(t) \Bigg(\mathbf{1}_{\{|M(\mathbf{H}(t-))|<k-1\}} \Bigg(\left(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-)) \right) \\
&\qquad\qquad \times \sum_{j\in I\setminus M(\mathbf{H}(t-))} \Big(C(t,\mathbf{H}^j(t-)) - C(t,\mathbf{H}(t-))\Big) \Big(h_j^{\mathbb{P}}(\mathbf{H}(t-)) - h_j(\mathbf{H}(t-))\Big) dt
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\sum_{j\in I\setminus M(\mathbf{H}(t-))}\Big(C(t,\mathbf{H}^j(t-))-C(t,\mathbf{H}(t-))\Big)d\xi_j^{\mathbb{P}}(t)\Bigg)\\
&+\mathbf{1}_{\{|M(\mathbf{H}(t-))|=k-1\}}\Bigg(\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\\
&\qquad\qquad\times\Bigg(\sum_{j\in I\setminus M(\mathbf{H}(t-))}\Big(R_j-C(t,\mathbf{H}(t-))\Big)\Big(h_j^{\mathbb{P}}(\mathbf{H}(t-))-h_j(\mathbf{H}(t-))\Big)\Bigg)dt\\
&-\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\sum_{j\in I\setminus M(\mathbf{H}(t-))}\left(R_j-C(t,\mathbf{H}(t-))\right)d\xi_j^{\mathbb{P}}(t)\Bigg)\Bigg)\\
=&rW_t dt\\
&+\mathbf{1}_{\{|M(\mathbf{H}(t-))|\le k-1\}}\phi(t)\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\\
&\qquad\qquad\times\sum_{j\in I\setminus M(\mathbf{H}(t-))}\Big(C(t,\mathbf{H}^j(t-))-C(t,\mathbf{H}(t-))\Big)\Big(h_j^{\mathbb{P}}(\mathbf{H}(t-))-h_j(\mathbf{H}(t-))\Big)dt\\
&+\mathbf{1}_{\{|M(\mathbf{H}(t-))|\le k-1\}}\phi(t-)\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\\
&\qquad\qquad\times\sum_{j\in I\setminus M(\mathbf{H}(t-))}\Big(C(t,\mathbf{H}^j(t-))-C(t,\mathbf{H}(t-))\Big)d\xi_j^{\mathbb{P}}(t).
\end{aligned}
$$

We can rewrite the dynamic with jumps,

$$
\begin{aligned}
&dW_t(\phi)\\
=&\Bigg(rW_t-\mathbf{1}_{\{|M(\mathbf{H}(t-))|\le k-1\}}\phi(t)\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\\
&\qquad\qquad\times\sum_{j\in I\setminus M(\mathbf{H}(t-))}\Big(C(t,\mathbf{H}^j(t-))-C(t,\mathbf{H}(t-))\Big)h_j(\mathbf{H}(t-))\Bigg)dt\\
&+\mathbf{1}_{\{|M(\mathbf{H}(t-))|\le k-1\}}\phi(t-)\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\\
&\qquad\qquad\times\sum_{j\in I\setminus M(\mathbf{H}(t-))}\Big(C(t,\mathbf{H}^j(t-))-C(t,\mathbf{H}(t-))\Big)dH_j(t). \qquad \text{(D–76)}
\end{aligned}
$$

Since

$$
\pi(t)=\frac{\phi(t)\widetilde{C}(t,\mathbf{H}(t))}{W^{\phi}(t)}\ ,
$$

we can obtain

$$
\begin{aligned}
&dW_t\\
=&\Bigg(rW_t + \mathbf{1}_{\{|M(\mathbf{H}(t-))|\le k-1\}}\pi_t W_t \sum_{j\in I\setminus M(\mathbf{H}(t-))}\left(\frac{C(t,\mathbf{H}^j(t-))}{C(t,\mathbf{H}(t-))}-1\right)\lambda_j h_j(\mathbf{H}(t-))\Bigg)dt\\
&+\mathbf{1}_{\{|M(\mathbf{H}(t-))|\le k-1\}}\pi_{t-}W_{t-}\sum_{j\in I\setminus M(\mathbf{H}(t-))}\left(\frac{C(t,\mathbf{H}^j(t-))}{C(t,\mathbf{H}(t-))}-1\right)d\xi_j^{\mathbb{P}}(t)\\
=&\Bigg(rW_t - \mathbf{1}_{\{|M(\mathbf{H}(t-))|\le k-1\}}\pi_t W_t \sum_{j\in I\setminus M(\mathbf{H}(t-))}\left(\frac{C(t,\mathbf{H}^j(t-))}{C(t,\mathbf{H}(t-))}-1\right)h_j(\mathbf{H}(t-))\Bigg)dt\\
&+\mathbf{1}_{\{|M(\mathbf{H}(t-))|\le k-1\}}\pi_{t-}W_{t-}\sum_{j\in I\setminus M(\mathbf{H}(t-))}\left(\frac{C(t,\mathbf{H}^j(t-))}{C(t,\mathbf{H}(t-))}-1\right)dH_j(t). \qquad \text{(D–77)}
\end{aligned}
$$

Step 2. The external shock does not realize.

$$
\begin{aligned}
&dW_t(\boldsymbol{\phi})\\
=&\phi(t-)\Big(d\widetilde{C}(t,\mathbf{H}(t)) + dD_t\Big) + \phi^B(t)dB_t\\
=&\phi(t-)\Bigg(\mathbf{1}_{\{|M(\mathbf{H}(t-))|<k-1\}}\Bigg(\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\left(rC(t,\mathbf{H}(t-))-\kappa\right)dt\\
&+\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\sum_{j\in I\setminus M(\mathbf{H}(t-))}\Big(C(t,\mathbf{H}^j(t-))-C(t,\mathbf{H}(t-))\Big)d\xi_j^{\mathbb{P}}(t)\\
&+\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\sum_{j\in I\setminus M(\mathbf{H}(t-))}\Big(C(t,\mathbf{H}^j(t-))-C(t,\mathbf{H}(t-))\Big)\\
&\qquad\qquad\times\Big(h_j^{\mathbb{P}}(\mathbf{H}(t-))-h_j(\mathbf{H}(t-))\Big)dt\\
&+\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\cup\{0\}\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\Big(C(t,\mathbf{H}^0(t-))-C(t,\mathbf{H}(t-))\Big)d\xi_0^{\mathbb{P}}(t)\\
&+\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\cup\{0\}\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\\
&\quad\times\Big(C(t,\mathbf{H}^0(t-))-C(t,\mathbf{H}(t-))\Big)\Big(h_0^{\mathbb{P}}(\mathbf{H}(t-))-h_0(\mathbf{H}(t-))\Big)dt\Bigg)\\
&+\mathbf{1}_{\{|M(\mathbf{H}(t-))|=k-1\}}\Bigg(\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\\
&\qquad\times\Big(rC(t,\mathbf{H}(t-))-\kappa-\sum_{j\in I\setminus M(\mathbf{H}(t-))}C(t,\mathbf{H}^j(t-))h_j(\mathbf{H}(t-))\Big)dt
\end{aligned}
$$

$$
\begin{aligned}
&-\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\sum_{j\in I\setminus M(\mathbf{H}(t-))} C(t,\mathbf{H}(t-))d\xi_j^{\mathbb{P}}(t)\\
&-\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\\
&\qquad\times\sum_{j\in I\setminus M(\mathbf{H}(t-))} C(t,\mathbf{H}(t-))\Big(h_j^{\mathbb{P}}(\mathbf{H}(t-))-h_j(\mathbf{H}(t-))\Big)dt\\
&+\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\cup\{0\}\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\Big(C(t,\mathbf{H}^0(t-))-C(t,\mathbf{H}(t-))\Big)d\xi_0^{\mathbb{P}}(t)\\
&+\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\cup\{0\}\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\\
&\qquad\times\Big(C(t,\mathbf{H}^0(t-))-C(t,\mathbf{H}(t-))\Big)\Big(h_0^{\mathbb{P}}(\mathbf{H}(t-))-h_0(\mathbf{H}(t-))\Big)dt\Bigg)\Bigg)\\
&+\phi(t-)\Bigg(\mathbf{1}_{\{|M(\mathbf{H}(t-))|\le k-1\}}\kappa\,dt+\mathbf{1}_{\{|M(\mathbf{H}(t-))|=k-1\}}\sum_{j\in I\setminus M(\mathbf{H}(t-))} R_j\,dH_j(t)\Bigg)\\
&+r\phi^B(t)B_t dt\\
=&\phi(t-)\Bigg(\mathbf{1}_{\{|M(\mathbf{H}(t-))|<k-1\}}\Bigg(\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\\
&\qquad\qquad\times\sum_{j\in I\setminus M(\mathbf{H}(t-))}\Big(C(t,\mathbf{H}^j(t-))-C(t,\mathbf{H}(t-))\Big)d\xi_j^{\mathbb{P}}(t)\\
&+\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\\
&\qquad\times\sum_{j\in I\setminus M(\mathbf{H}(t-))}\Big(C(t,\mathbf{H}^j(t-))-C(t,\mathbf{H}(t-))\Big)\Big(h_j^{\mathbb{P}}(\mathbf{H}(t-))-h_j(\mathbf{H}(t-))\Big)dt\\
&+\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\cup\{0\}\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\Big(C(t,\mathbf{H}^0(t-))-C(t,\mathbf{H}(t-))\Big)d\xi_0^{\mathbb{P}}(t)\\
&+\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\cup\{0\}\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\\
&\qquad\times\Big(C(t,\mathbf{H}^0(t-))-C(t,\mathbf{H}(t-))\Big)\Big(h_0^{\mathbb{P}}(\mathbf{H}(t-))-h_0(\mathbf{H}(t-))\Big)dt\Bigg)\\
&+\mathbf{1}_{\{|M(\mathbf{H}(t-))|=k-1\}}\Bigg(\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&\times\Big(-\sum_{j\in I\setminus M(\mathbf{H}(t-))} C(t,\mathbf{H}^j(t-))h_j(\mathbf{H}(t-))\Big)dt\\
&-\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\sum_{j\in I\setminus M(\mathbf{H}(t-))}\Big(R_j-C(t,\mathbf{H}(t-))\Big)d\xi_j^{\mathbb{P}}(t)\\
&+\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\sum_{j\in I\setminus M(\mathbf{H}(t-))} R_j h_j^{\mathbb{P}}(\mathbf{H}(t-))dt\\
&-\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\\
&\quad\times\sum_{j\in I\setminus M(\mathbf{H}(t-))} C(t,\mathbf{H}(t-))\Big(h_j^{\mathbb{P}}(\mathbf{H}(t-))-h_j(\mathbf{H}(t-))\Big)dt\\
&+\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\cup\{0\}\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\Big(C(t,\mathbf{H}^0(t-))-C(t,\mathbf{H}(t-))\Big)d\xi_0^{\mathbb{P}}(t)\\
&+\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\cup\{0\}\setminus M(\mathbf{H}(t-))}} H_m(t)(1-H_n(t-))\Bigg)\\
&\quad\times\Big(C(t,\mathbf{H}^0(t-))-C(t,\mathbf{H}(t-))\Big)\Big(h_0^{\mathbb{P}}(\mathbf{H}(t-))-h_0(\mathbf{H}(t-))\Big)dt\Bigg)\Bigg)\\
&+r\phi(t)CLN_t dt+r\phi^B(t)B_t dt\\
=&rW_t dt\\
&+\phi(t)\Bigg(\mathbf{1}_{\{|M(\mathbf{H}(t-))|<k-1\}}\Bigg(\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\\
&\qquad\times\sum_{j\in I\setminus M(\mathbf{H}(t-))}\Big(C(t,\mathbf{H}^j(t-))-C(t,\mathbf{H}(t-))\Big)\Big(h_j^{\mathbb{P}}(\mathbf{H}(t-))-h_j(\mathbf{H}(t-))\Big)dt\\
&+\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\sum_{j\in I\setminus M(\mathbf{H}(t-))}\Big(C(t,\mathbf{H}^j(t-))-C(t,\mathbf{H}(t-))\Big)d\xi_j^{\mathbb{P}}(t)\\
&+\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\cup\{0\}\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\Big(C(t,\mathbf{H}^0(t-))-C(t,\mathbf{H}(t-))\Big)d\xi_0^{\mathbb{P}}(t)\\
&+\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\cup\{0\}\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\\
&\quad\times\Big(C(t,\mathbf{H}^0(t-))-C(t,\mathbf{H}(t-))\Big)\Big(h_0^{\mathbb{P}}(\mathbf{H}(t-))-h_0(\mathbf{H}(t-))\Big)dt\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+ \mathbf{1}_{\{|M(\mathbf{H}(t-))|=k-1\}}\Bigg(\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg) \\
&\qquad\qquad \times \Bigg(\sum_{j\in I\setminus M(\mathbf{H}(t-))}\Big(R_j - C(t,\mathbf{H}(t-))\Big)\Big(h_j^{\mathbb{P}}(\mathbf{H}(t-)) - h_j(\mathbf{H}(t-))\Big)\Bigg)dt \\
&- \Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\sum_{j\in I\setminus M(\mathbf{H}(t-))}\Big(R_j - C(t,\mathbf{H}(t-))\Big)d\xi_j^{\mathbb{P}}(t) \\
&+ \Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\cup\{0\}\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\Big(C(t,\mathbf{H}^0(t-)) - C(t,\mathbf{H}(t-))\Big)d\xi_0^{\mathbb{P}}(t) \\
&+ \Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\cup\{0\}\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg) \\
&\qquad \times \Big(C(t,\mathbf{H}^0(t-)) - C(t,\mathbf{H}(t-))\Big)\Big(h_0^{\mathbb{P}}(\mathbf{H}(t-)) - h_0(\mathbf{H}(t-))\Big)dt\Bigg)\Bigg) \\
=& rW_t dt \\
&+ \mathbf{1}_{\{|M(\mathbf{H}(t-))|\le k-1\}}\Bigg(\phi(t)\Bigg(\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg) \\
&\qquad \times \sum_{j\in I\setminus M(\mathbf{H}(t-))}\Big(C(t,\mathbf{H}^j(t-)) - C(t,\mathbf{H}(t-))\Big)\Big(h_j^{\mathbb{P}}(\mathbf{H}(t-)) - h_j(\mathbf{H}(t-))\Big) \\
&\qquad + \Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\cup\{0\}\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg) \\
&\qquad\qquad \times \Big(C(t,\mathbf{H}^0(t-)) - C(t,\mathbf{H}(t-))\Big)\Big(h_0^{\mathbb{P}}(\mathbf{H}(t-)) - h_0(\mathbf{H}(t-))\Big)\Bigg)dt \\
&+ \phi(t-)\Bigg(\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg) \\
&\qquad\qquad \times \sum_{j\in I\setminus M(\mathbf{H}(t-))}\Big(C(t,\mathbf{H}^j(t-)) - C(t,\mathbf{H}(t-))\Big)d\xi_j^{\mathbb{P}}(t) \\
&\qquad + \Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\cup\{0\}\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg) \\
&\qquad\qquad \times \Bigg(C(t,\mathbf{H}^0(t-)) - C(t,\mathbf{H}(t-))\Bigg)d\xi_0^{\mathbb{P}}(t)\Bigg)\Bigg).
\end{aligned}
$$

The dynamic process with jumps is as follows,

$$dW_t(\phi)$$

$$
\begin{aligned}
=&\Bigg(rW_t - \mathbf{1}_{\{|M(\mathbf{H}(t-))|\le k-1\}}\phi(t)\Bigg(\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\\
&\qquad\times\sum_{j\in I\setminus M(\mathbf{H}(t-))}\Big(C(t,\mathbf{H}^j(t-))-C(t,\mathbf{H}(t-))\Big)h_j(\mathbf{H}(t-))\\
&\qquad+\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\cup\{0\}\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\\
&\qquad\times\Big(C(t,\mathbf{H}^0(t-))-C(t,\mathbf{H}(t-))\Big)h_0(\mathbf{H}(t-))\Bigg)\Bigg)dt\\
&+\mathbf{1}_{\{|M(\mathbf{H}(t-))|\le k-1\}}\phi(t-)\Bigg(\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\\
&\qquad\times\sum_{j\in I\setminus M(\mathbf{H}(t-))}\Big(C(t,\mathbf{H}^j(t-))-C(t,\mathbf{H}(t-))\Big)dH_j(t)\\
&\qquad+\Bigg(\prod_{\substack{m\in M(\mathbf{H}(t-))\\ n\in I\cup\{0\}\setminus M(\mathbf{H}(t-))}} H_m(t-)(1-H_n(t-))\Bigg)\\
&\qquad\times\Big(C(t,\mathbf{H}^0(t-))-C(t,\mathbf{H}(t-))\Big)dH_0(t)\Bigg).
\end{aligned}
\tag{D–78}
$$

So, we have,

$$
\begin{aligned}
&dW_t(\pi)\\
=&\Bigg(rW_t-\mathbf{1}_{\{|M(\mathbf{H}(t-))|\le k-1\}}\pi(t)W_t\Bigg(\sum_{j\in I\setminus M(\mathbf{H}(t-))}\left(\frac{C(t,\mathbf{H}^j(t-))}{C(t,\mathbf{H}(t-))}-1\right)h_j(\mathbf{H}(t-))\\
&\qquad+\left(\frac{C(t,\mathbf{H}^0(t-))}{C(t,\mathbf{H}(t-))}-1\right)h_0(\mathbf{H}(t-))\Bigg)\Bigg)dt\\
&+\mathbf{1}_{\{|M(\mathbf{H}(t-))|\le k-1\}}\pi(t-)W_{t-}\Bigg(\sum_{j\in I\setminus M(\mathbf{H}(t-))}\left(\frac{C(t,\mathbf{H}^j(t-))}{C(t,\mathbf{H}(t-))}-1\right)dH_j(t)\\
&\qquad+\left(\frac{C(t,\mathbf{H}^0(t-))}{C(t,\mathbf{H}(t-))}-1\right)dH_0(t)\Bigg).
\end{aligned}
\tag{D–79}
$$

By the further calculation, we can obtain (19).